2010 LSU Math Contest
Open Session

Questions 1 - 13 are worth 1 point each and questions 14 - 25 are worth 2 points each.

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

You have one hour and twenty minutes to complete the entire morning exam.

Questions 1 - 13 Multiple Choice

Please:
- *Use the answer sheet for your answers.*
- *Answer only one choice A, B, C, D, or E for each question by circling your answer on the answer sheet.*
- *Completely erase any answer you wish to change.*
- *Do not make stray marks on the answer sheet.*

1
Subtracting 99% of 19 from 19% of 99,
the difference d satisfies

A $d < -1$ B $d = -1$ C $-1 < d < 1$ D $d = 1$ E $d > 1$

2
If the lateral surface of a cylinder when unfolded is a square of side 1, then

A The area of the base of the cylinder is the same number as is the volume.

B It is possible to construct a sphere tangent at one point to the top, at one point to the base, and along an equator to the lateral surface of the cylinder.

C The volume of this cylinder is π.

D

E Such a cylinder does not exist

3
Which of the following statements about the number $\sin(x^2)$ is true?

A It is equal to $\sqrt{1 - \cos^2(x^2)}$ for any x.

B It is equal to $(\sin x)^2$ for any x.

C It can assume any values from the closed interval $[-1, 1]$.

D It is non-negative for any $x \geq 0$.

E None of the above is true.

4
$\log_{0.25} 128 =$

A $\frac{7}{2}$ B $-\frac{7}{2}$ C $\frac{2}{7}$ D 14 E -14

5
We are solving for $x \in [0, 4\pi]$ the equation $\sin x = p$, where p is a fixed real number (a parameter).
Which of the following statements is always false, regardless of what p might be?

A This equation has precisely 2 solutions.

B This equation has precisely 3 solutions.

C This equation has precisely 4 solutions.

D This equation has precisely 5 solutions.

E This equation has no solutions.

6
Let a, b be positive numbers not equal to 1. Then $\dfrac{\log_a(b^2)}{\log_{a^2} b} =$

A 1/2 B 1 C 2 D 4 E 8

7
$\sin^2 75° - \cos^2 75° =$

A $-\frac{\sqrt{3}}{2}$ B $\frac{\sqrt{3}}{2}$ C $\frac{1}{2}$ D $-\frac{1}{2}$ E 1

8
How many four digit numbers less than 4321 and made of digits 1, 2, 3, 4 are there?

A 128 B 192 C 228 D 311 E 4320

9
If $\cos \alpha = \dfrac{1}{3}$ with $\alpha \in \left(-\dfrac{\pi}{2}, 0\right)$, then $\tan \alpha =$

A $2\sqrt{2}$ B $\sqrt{2}$ C $-\frac{\sqrt{2}}{4}$ D $-\sqrt{2}$ E $-2\sqrt{2}$

10
A circle is inscribed in a square of side one, and a right triangle is inscribed in the circle. The sum of perimeters of all the three shapes

A can be smaller than $6 + \pi$

B can be equal to $4 + \pi + 2.1$

C can be equal to 10

D can be equal to 12

E none of the above

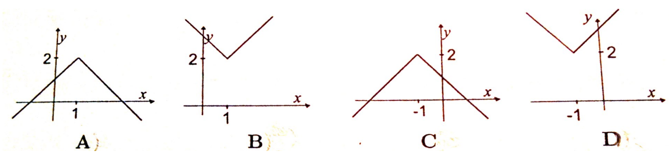
11
Let a, b, c and d be the roots of the equation
$x^4 - 8x^3 - 21x^2 + 148x - 160 = 0$.
The value of $\frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}$ is

A $-\dfrac{4}{37}$ B $-\dfrac{1}{20}$ C $\dfrac{1}{20}$ D $\dfrac{4}{37}$ E none of these

12
The real numbers x, y and z satisfy $2^{x+y} = 10$, $2^{y+z} = 20$ and $2^{z+x} = 30$. Then 2^x is

A $\dfrac{\sqrt{6}}{2}$ B $\dfrac{3}{2}$ C $\sqrt{15}$ D $10\sqrt{6} - 20$ E none of these

13
Which of the following



E None of the above

is the graph of $f(x) = -|x-1| + 2$?

Questions 14 - 25 Exact Answers

These next twelve questions require exact numerical or algebraic answers. Hand written exact answers must be written on the answer sheet with fractions reduced, radicals simplified, and denominators rationalized (Improper fractions can be left alone or changed to mixed fractions). Do not make an approximation for π or other irrational numbers. Answers must be exact. Large numbers should not be multiplied out, i.e., do not try to multiply out $20!$ or 6^{40}.

14 Find $\sin\left(\dfrac{2010}{6}\pi\right)$.

15 Find all the solutions to the equation

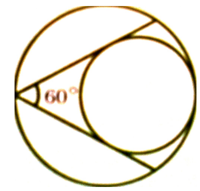
$$\left(\frac{2x^2-5}{3}\right)^{x^2-3x} = 1 .$$

16 let f be a function which satisfies $f(29+x) = f(29-x)$ for all real numbers x. If the equation $f(x) = 0$ has exactly three distinct real solutions α, β, and γ, determine the value of $\alpha + \beta + \gamma$.

17 There is a cube of side length 4. It is filled with spheres of diameter 2, each of which is tangent three sides of the cube, and to three other spheres. Assuming that no sphere goes outside the cube, what is the probability that a spot chosen randomly inside the cube will lie in one of the spheres?

18 How many integers between 10 and 99 are there which have the units digit greater than the tens digit?

19 In the diagram on the right, the bigger circle has area 1 in^2.
What is the area of the smaller circle?
(the marked angle is $60°$)

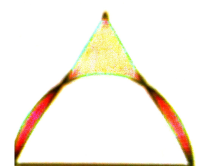


20 Find all the solutions to the equation

$$||||x-1|-2|-3|-4| = 0 .$$

21 Let a and b be two different real numbers such that $a^2 = 6b + 5ab$ and $b^2 = 6a + 5ab$. Find ab.

22 In the diagram on the right, the triangle with diameter of a semicircle as a base is equilateral. The semicircle has radius 1.
Find the shaded area.
(i.e. the area outside of the triangle and inside the circle on the sides plus the area outside the circle and inside the triangle on top.)



23 For a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x (for example $\lfloor -1.5 \rfloor = -2$), and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Determine all real numbers x for which $\lfloor x \rfloor \cdot \{x\} = x$.

24 In a geometric sequence of real numbers, the sum of the first two terms is 7 and the sum of the first six terms is 91. What is the sum of the first four terms?

25 A list of six positive integers p, q, r, s, t, u satisfies $p < q < r < s < t < u$. There are exactly 15 pairs of numbers that can be formed by choosing two different numbers from this list. The sums of these 15 pairs of numbers are: 25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117.
Which sum equals $r + s$?

Tie Breaker requiring Full Solution

Please give a **detailed explanation** on the answer sheet **of** your **solution to Question 25**.

This tie breaker question is graded as an essay question, i.e. it is graded for the clarity of explanation and argument as well as correctness.

It is the only question graded for partial credit. Do not hesitate to write your thoughts even if your solution is not rigorous!

It is graded only to separate first, second, and third place ties.