2008 LSU Math Contest
Team Problems

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

You have one hour and fifteen minutes to complete the entire team session.

These 10 questions require exact numerical or algebraic answers. Hand written exact answers must be written with fractions reduced, radicals simplified, and denominators rationalized. Do not make an approximation for π or other irrational numbers. Answers must be exact.

The tiebreaker for the team competition is time. *If your team reaches a point where you are satisfied or expect that you will not have more solutions in the allotted time, then you may wish to turn in your paper a little early to get a time advantage.*

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1 $\triangle ABC$ with $AB = AC = 5$, $BC = 6$ is inside the circle ABC. Find the radius r of the *circumcircle* ABC.

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2 Find all the integers between 100 and 1000 which are equal to the square of some integer and also equal to the cube of some other integer.

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3 List the numbers 1 through 60 (in order, with no spaces) to make a new number. Cross out all but 9 of the digits to ceate the largest number possible, without rearranging the digits.

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4 Consider n distinct complex numbers $z_i \in \mathbf{C}$ such that

$$\min_{i \neq j} |z_i - z_j| \geq \max_{i \leq n} |z_i|.$$

What is the greatest possible value of n?

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5 Consider a polynomial $f(x)$ with real coefficients having the property $f(g(x)) = g(f(x))$ for every polynomial $g(x)$ with real coefficients. Determine $f(x)$.

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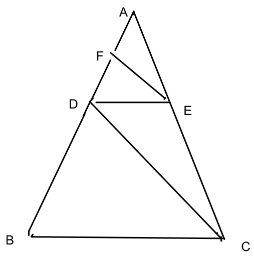
6 In $\triangle ABC$, $\overline{DE} \parallel \overline{BC}$, $\overline{FE} \parallel \overline{DC}$, $AF = 4$, and $FD = 6$. Find DB.



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7 Cucumbers are assumed, for present purposes, to be a substance that is 99% water by weight. If 500 pounds of cucumbers are allowed to stand overnight, and if the partially evaporated substance that remains in the morning is 98% water, how much is the morning weight?

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8 Let x be a real number such that $\sec x - \tan x = 2$. Evaluate $\sec x + \tan x$.

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9 Simplify

$$\left(\frac{1 \cdot 2 \cdot 4 + 2 \cdot 4 \cdot 8 + \cdots + n \cdot 2n \cdot 4n}{1 \cdot 3 \cdot 9 + 2 \cdot 6 \cdot 18 + \cdots + n \cdot 3n \cdot 9n} \right)^{1/3}.$$

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10 A king is placed on left bottom square $a1$ of a 4×4 chessboard with rows 1, 2, 3, 4 (from bottom to the top) and with columns $a,\ b,\ c,\ d$ (from left to right). Two players take turns moving the king either upwards, to the right, or along a diagonal going upwards and to the right. The player who places the king on the top right square $d4$ is the winner and the square $d4$ is a winning position. More generally, we call a square of the chessboard *a winning position* if

(1) The square is the final position of the game;

(2) A player can never move from one winning position to another in a single turn;

(3) A player can always move from a non-winning position to a winning one in a single move.

The discovery of such a class of all winning positions for a given game is equivalent to solving the game, because moving to a winning position at each move constitues a winning strategy. If the initial position of the game is a winning one, then the second player will win. Otherwise, the first player will win.
Find the set of all winning positions for this game.
