2008 LSU Math Contest
Open Session

Questions 1 - 12 are worth 1 point each and questions 13 - 24 are worth 2 points each.

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

You have one hour and twenty minutes to complete the entire morning exam.

Questions 1 - 12 Multiple Choice

Please:
- *Use the answer sheet for your answers.*
- *Answer only one choice A, B, C, D, or E for each question by circling your answer on the answer sheet.*
- *Completely erase any answer you wish to change.*
- *Do not make stray marks on the answer sheet.*

1

The vertex of the parabola defined by $x^2 + 4x + 2y + 10 = 0$ is located at

A $(2, 3)$ B $(-3, 2)$ C $(-2, -3)$ D $(2, 5)$ E $(5, 2)$

2

$\sqrt{\pi}\sqrt[3]{\pi} =$

A π^{-5} B π^{5} C $\pi^{\frac{1}{5}}$ D $\pi^{\frac{5}{6}}$ E $\pi^{\frac{1}{6}}$

3

Which of the three following definitions are functions of x?
Take a number x between 0 and 90, draw a right triangle ABC with angle A measuring x degrees and call the opposite side a.
Let $f(x)$ be the length of the hypotenuse,
Let $g(x)$ be the length of the hypotenuse measured in inches and
Let $h(x)$ be the length of the hypotenuse divided by the length of side a.

A All of f, g and h are functions.

B Only f and g are functions.

C Only g is a function.

D Only h is a function.

E None of the above is correct.

4

Which of the following statements are true?

(I): $-1 < x < 2$ implies $0 < x^2 < 4$

(II): $-1 < x < 2$ implies $1 < x^2 < 4$

(III): $-1 < x < 2$ implies $-1 < x^2 < 4$

A Only I. B Only II. C Only III.

D Both I and II. E All of I, II, and III.

5

Suppose that θ is a number and $0 < \theta < 90$. Consider the following statements and select the correct choice.

1. $\sec(\theta^\circ) < 1$ 2. $\sec(\theta^\circ) = 1$ 3. $\sec(\theta^\circ) > 1$

A Only 1 is possible.B All are possible.C Only 3 is possible.
D Only 1 is impossible. E Only 3 is impossible.

6

Which of the following is true about base 10 logarithms?

A $\log(x + 1) = \log x + \log 1$ for all $x > 0$.

B $(\log x)(\log 1) = 0$ for all $x > 0$.

C $\frac{\log(x+2)}{\log(x-1)} = \log(x + 2) - \log(x - 1)$ for all $x > 1$.

D $10^x = \frac{1}{\log x}$.

E More than one of the above choices are true.

7

The diameter of one circle equals the radius of a second circle. Find the ratio of their areas.

A 1:2 B 1:3 C 1:4 D 1:5 E 1:8

8

Let m and n be two positive integers. Consider the following two statements about m and n:

(I) The number $m - n$ is odd.

(II) The number $m^2 - n^2$ is odd.

Which of the following statements is true?

A (I) is necessary but not sufficient for (II)

B (I) is sufficient but not necessary for (II)

C (I) is neither necessary nor sufficient for (II)

D (I) is necessary and sufficient for (II)

E None of the above

9

Suppose that $\log_2(\log_3(\log_5(\log_7 N))) = 11$.
How many different prime numbers are factors of N ?

A 1 B 2 C 3 D 4 E 7

10

The three positive real numbers x, y, z satisfy the system of two equations:

$$\frac{z}{x+y} = 2, \qquad \frac{z}{x-y} = 3.$$

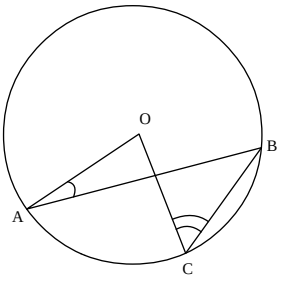
Which of the following is true:

A $x < y < z$ B $y < z < x$ C $z < x < y$
D $y < x < z$ E $x < z < y$

11

In the diagram, O is the center of the circle, $\angle OAB = 10°$ and $\angle OCB = 30°$.

Then $\angle ABC =$



A $10°$ B $20°$ C $30°$ D $40°$ E $50°$

12

The following 'proof' that $1 = 2$ of course contains a mathematical error.
Which line (A, B, C, D or E) *is not* a consequence of the previous one?

Let $a = b$
A $ab = b^2$
B $ab - a^2 = b^2 - a^2$
C $a(b-a) = (b+a)(b-a)$
D $a = b + a$
E when $a = 1$ and $b = 1$, we get $1 = 1 + 1 = 2$.

Questions 13 - 24 Exact Answers

These next twelve questions require exact numerical or algebraic answers. Hand written exact answers must be written on the answer sheet with fractions reduced, radicals simplified, and denominators rationalized (Improper fractions can be left alone or changed to mixed fractions). Do not make an approximation for π or other irrational numbers. Answers must be exact. Large numbers should not be multiplied out, i.e., do not try to multiply out $20!$ *or* 6^{40}.

13 Suppose r is a root of the polynomial

$$x^4 - x^3 + x^2 - x + 1.$$

What is $r^{20} - r^{15} + r^{10} - r^5 + 1$?

14 What is the numerical value of

$$\log_2(3)\log_3(4)\log_4(5)\log_5(6)\log_6(7)\log_7(8).$$

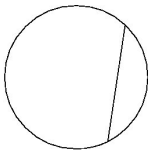
15 In the triangle $\triangle ABC$, the measure of $\angle A$ is twice the measure of $\angle C$. Also the length of $\overline{AB}$ is 2 and the length of $\overline{BC}$ is 3. Find the length of $\overline{AC}$.

16 How many real roots at most can the equation

$$x|x| + px + q = 0$$

have?

17 If the figure on the right is a circle of radius 1 unit, and the chord has a length of one unit, what is the area of the smaller region between the chord and the circle?



18 Suppose A and B are points on the circle with center C. The angle $\angle ACB$ is $15°$. If a point D is randomly chosen on the circle, then what is the probability that the triangle $\triangle ABD$ is *obtuse*?

19 Three people take turns drawing a card from a shuffled deck, returning the card and reshuffling after each draw. They agree that whoever draws a heart first will pay for dinner. If in the first set of drawings no one draws a heart, they draw again, taking turns in the same order, and they continue in this manner until eventually someone draws a heart. What is the probability that the third person will have to pay for dinner?

20 If $a679b$ is a five digit number (in base 10) which is divisible by 72, determine a and b.

21 Two circles of radius 2 lie in the plane. Line ℓ_1 is tangent to both circles with the circles on different sides of ℓ_1, while line ℓ_2 is tangent to both circles with both circles on the same side of ℓ_2. The angle between lines ℓ_1 and ℓ_2 is $30°$. Find the distance between the centers of the circles.

22 How many different positive integers are divisors of 1,000,000,000,000?

23 Let f be a function with the following properties:

1) $f(n)$ is defined for every positive integer n;
2) $f(n)$ is an integer;
3) $f(2) = 2$;
4) $f(mn) = f(m)f(n)$ for all m and n;
5) $f(m) > f(n)$ whenever $m > n$.

Find $f(2008)$.

24 Let $f(x) = x^2 + ax + b$, where a and b are two real numbers, whose exact values are not known yet. Question: Must the largest of $|f(1)|, |f(2)|, |f(3)|$ be at least 1/2?

Tie Breaker requiring Full Solution

Please give a detailed explanation on the answer sheet to your solution to Question **24** above.

This tie breaker question is graded as an essay question, i.e. it is graded for the clarity of explanation and argument as well as correctness.

It is the only question graded for partial credit.

It is graded only to separate first, second, and third place ties.