2006 LSU Math Contest
Team Problems

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

You have one hour and fifteen minutes to complete the entire team session.

These 10 questions (except for questions 3 and 9) require exact numerical or algebraic answers. Hand written exact answers must be written with fractions reduced, radicals simplified, and denominators rationalized. Do not make an approximation for π or other irrational numbers. Answers must be exact.

The tiebreaker for the team competition is time. *If your team reaches a point where you are satisfied or expect that you will not have more solutions in the allotted time, then you may wish to turn in your paper a little early to get a time advantage.*

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1 Find all the solutions x, y in digits 1 – 9 of

$$\begin{array}{r} x \\ yyyy \\ yyyy \\ yyyy \\ \underline{+yyyy} \\ xyyyy \end{array}$$

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2 A Sultan had many children. His oldest son was a twin. Moreover, all his children, except for 41 of them, were twins. Also, all his children except for 41 of them were triplets. And finally, all his children, except for 41 of them, were quadruplets. How many children did the Sultan have? (Quadruplets are not counted as triplets, etc.)

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3 Which of the following two finite positive numbers (where 2 is repeated infinitely many times in each) is larger:

$$\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}} \text{ or } \sqrt{2\sqrt{2\sqrt{2\ldots}}}\,?$$

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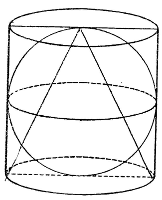
4 A sphere and a cone are inscribed in a cylinder as shown



Let $V_{\bigcirc}, V_{\triangle}, V_{\square}$ denote respectively their volumes. Find the double ratio $V_{\triangle} : V_{\bigcirc} : V_{\square}$.

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5 P is a point on the plane of the square $ABCD$ such that each of PAB, PBC, PCD and PDA is an isosceles (nondegenerate) triangle. How many possible positions are there for such a point P?

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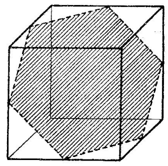
6 A hexagon is inscribed in a cube of side a as shown with the vertices of the hexagon being the centers of the corresponding edges of the cube.



Find the area of the hexagon. .

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7 What is the general solution of $\cos x = \cos \frac{x}{4}$ and how many distinct solutions are there in the interval $[0, 8\pi]$?

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8 Give a counterexample to disprove the following statement:

Let a and b be positive real numbers different from 1. Then $\log_a b + \log_b a \geq 2$.

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9 A function $f : \mathbb{R} \to \mathbb{R}$ is such that $f(\frac{1-x}{1+x}) = x$ for all $x \neq -1$. Which of the following statements are true:

A $f(f(x)) = -x$ B $f(\frac{1+x}{1-x}) = f(-x), x \neq 1$

C $f(\frac{1}{x}) = f(x), x \neq 0$ D $f(-x-2) = -f(x) - 2$

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10 [This problem was invented by Edouard Lucas, a French nineteenth century mathematician.]
Every day at noon† a ship leaves Le Havre in France for New York and another ship leaves New York for Le Havre. The trip lasts 7 days and 7 nights. How many New York-Le Havre ships will the ship leaving Le Havre today meet during its journey to New York (including the ships it might meet at the harbors)?
†) All schedules are on Greenwich Mean Time.
Hint: Try to make a schedule diagram.
