Team Problems

These questions require exact numerical or algebraic answers. Hand written exact answers must be written with fractions reduced, radicals simplified, and denominators rationalized. Do not make an approximation for π or other irrational numbers. Answers must be exact.

The tiebreaker for the team competition is time. If your team reaches a point where you are satisfied or expect that you will not have more solutions in the allotted time, then you may wish to turn in your paper a little early to get a time advantage.

No calculators, books, notes, cell phones... please!

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1. Let x_1 and x_2 be the real roots of the equation

$$x^2 - (k-2)x + (k^2 + 3k + 5) = 0.$$

What is the maximum possible value of $x_1^2 + x_2^2$ as k runs over the real numbers ?

2. Three cubes of volume 1, 8 and 27 are glued together at their faces. What is the smallest possible surface area of the resulting configuration ?

3. Find all the prime numbers the sum of whose digits is 4 and none of whose digits is 0.

4. [Henry Dudeney's puzzle] Inside a rectangular room 30 feet long, 12 feet wide, and 12 feet high, a spider eyes his supper. The spider is located on one end wall, halfway between the side walls, 1 foot from the ceiling. The unsuspecting fly rests directly opposite on the other end wall, 1 foot from the floor. What is the shortest distance the spider must crawl to get his supper? (He must crawl the entire way - no webs, please!)

5. Find all the integer solutions to the equation

$$2\,|\,x| + (-1)^x = 17.$$

6. Which number is greater: $\sqrt{2003} + \sqrt{2005}$ or $2\sqrt{2004}$?

7. Simplify the expression $\sqrt{4+\sqrt{7}} - \sqrt{4-\sqrt{7}}$ as much as you can.

8. Compute $49999989^2 - 50000011^2$.

9. A deck of 16 cards contains the four aces, four kings, four queens and four jacks. The 16 cards are thoroughly shuffled and my opponent (who always tells the truth) draws two cards simultaneously and at random from the deck. He says "I hold at least one ace". What is the chance that he holds two aces in his hand?

10. Simplify the fraction

$$\frac{x^8 + x^6 + x^4 + x^2 + 1}{x^4 + x^3 + x^2 + x + 1}.$$

11. The sides of a triangle are a, b, and $\sqrt{a^2 + ab + b^2}$. Find its largest angle.

12. Solve for x:

$$(5/22)^{3x-1} = 4.4^{2x-1}.$$

13. A watermelon, 99% of which is water, was placed in the sun. After a whole day 98% of the water evaporated. What fraction of the original watermelon is left?