1. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 4 & 1 & 5 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 3 & 4 \end{bmatrix}$.

(a) Calculate the determinant of A.

$$\det(A) = \begin{vmatrix} 1 & 1 & 0 & 3 \\ 0 & 4 & 1 & 5 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 3 \\ 0 & 4 & 1 & 5 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 3 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 0 & 3 \\ 0 & 4 & 1 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 24$$

For each of the following questions, give brief answers and explanations.

(b) Can the answer to part (a) be used to determine if the matrix A is invertible?
Yes, since $\det(A) \neq 0$, A is invertible.

(c) Can the answer to (a) be used to say how many solutions the system of equations $A\vec{x} = \vec{0}$ has?
Yes, since $\det(A) \neq 0$ (so A is invertible), $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = A^{-1}\vec{0} = \vec{0}$.

2. Recall that $\mathbb{R}^n$ denotes the vector space of all ordered n-tuples of real numbers.
Consider the vectors $\vec{v}_1 = (1, 2, 1)$, $\vec{v}_2 = (-1, 0, 1)$, $\vec{v}_3 = (2, 7, 3)$ in $\mathbb{R}^3$.

(a) Can the vector $\vec{b} = (3, 6, -1)$ be expressed as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$?

In matrix form, $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}$ is $A\vec{x} = \vec{b}$: $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 7 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & 0 & 7 & 6 \\ 1 & 1 & 3 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 1 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

From here, it is easy to see that $\text{rank}(A) = \text{rank}(A^\sharp) = 3$, so the system is consistent, and $\vec{b}$ can be expressed as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

(b) Does the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ form a basis for $\mathbb{R}^3$? Explain.
Yes. The calculation in (a) shows that any $\vec{b}$ in $\mathbb{R}^3$ can be expressed as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ uniquely (since the matrix A has rank 3 and so is invertible). Consequently, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $\mathbb{R}^3$. Furthermore, since $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$, the columns of A, i.e., $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, form a linearly independent set.

3. Let S be the subset of $\mathbb{R}^4$ consisting of vectors of the form $(s, t, t, s+t)$, where s and t are real numbers.

That is $S = \Big\{(s, t, t, s+t) \text{ in } \mathbb{R}^4 \ : \ s, t \text{ are real}\Big\} = \left\{ \begin{bmatrix} s \\ t \\ t \\ s+t \end{bmatrix} \text{ in } \mathbb{R}^4 \ : \ s, t \text{ are real} \right\}$.

(a) Show that S is a subspace of $\mathbb{R}^4$.
Let $\vec{u}$ and $\vec{v}$ be in S. Then $\vec{u} = (a, b, b, a+b)$ and $\vec{v} = (c, d, d, c+d)$ for some real numbers a, b, c, d. Since $\vec{u} + \vec{v} = (a+c, b+d, b+d, a+b+c+d) = (s, t, t, s+t)$ for the real numbers $s = a+c$ and $t = b+d$, $\vec{u} + \vec{v}$ is in S, and S is closed under vector addition.
If k is a scalar, and $\vec{u}$ is in S as above, then $k\vec{u} = (ka, kb, kb, ka+kb) = (s, t, t, s+t)$ for the real numbers $s = ka$ and $t = kb$. So $k\vec{u}$ is in S, and S is closed under scalar multiplication.

(b) Find a basis for S, and determine the dimension of S.
Since $(s,t,t,s+t) = s(1,0,0,1) + t(0,1,1,1)$, the set $\{(1,0,0,1),(0,1,1,1)\}$ spans S. It is easy to check that this set is linearly independent (do it). Consequently, $\{(1,0,0,1),(0,1,1,1)\}$ is a basis for S, and $\dim(S) = 2$.

4. Recall that P_3 denotes the vector space of all polynomials of degree at most 3 with real coefficients. Consider the polynomials $p_1 = 1+x^2+x^3$, $p_2 = 1+x+x^3$, $p_3 = x+x^2+x^3$ in P_3.

(a) Determine if the set $\{p_1,p_2,p_3\}$ is linearly independent.
If $c_1(1+x^2+x^3)+c_2(1+x+x^3)+c_3(x+x^2+x^3) = 0$, then
$(c_1+c_2)+(c_2+c_3)x+(c_1+c_3)x^2+(c_1+c_2+c_3)x^3 = 0+0x+0x^2+0x^3$.
For this to hold, we must have $c_1+c_2 = 0 \quad c_2+c_3 = 0 \quad c_1+c_3 = 0 \quad c_1+c_2+c_3 = 0$.
This is a homogeneous system of 4 linear equations in the 3 variables c_1, c_2, c_3.
Solving this system (do it) yields a unique solution: $c_1 = 0,\ c_2 = 0,\ c_3 = 0$.
Consequently, the set $\{p_1,p_2,p_3\}$ is linearly independent.

(b) Explain why the set $\{p_1,p_2,p_3\}$ does **not** span P_3.
For instance, since $\dim(P_3) = 4$ ($\{1,x,x^2,x^3\}$ is a basis), any spanning set must have at least 4 elements. Alternatively, you can check that the polynomial $a+bx+cx^2+dx^3$ is in $\operatorname{span}\{p_1,p_2,p_3\}$ only if $a+b+c-2d = 0$.

5. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 3 & -6 \end{bmatrix}$.

(a) Find a basis for the subspace of $\mathbb{R}^2$ spanned by the columns of A.
Give a brief geometric description of this subspace.
Write $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$. Since $\vec{v}_2 = -\vec{v}_1$ and $\vec{v}_3 = 2\vec{v}_1$, we have $\operatorname{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\} = \operatorname{span}\{\vec{v}_1\}$. So $\{\vec{v}_1\}$ is a basis. $\operatorname{span}\{\vec{v}_1\}$ is a line through the origin in $\mathbb{R}^2$.

(b) Recall that the nullspace of an $m \times n$ matrix A with real entries is the subspace of $\mathbb{R}^n$ defined by $\operatorname{nullspace}(A) = \{\vec{x} \text{ in } \mathbb{R}^n \ :\ A\vec{x} = \vec{0}\}$.
For the 2×3 matrix A above, find the dimension of the nullspace of A.
Give a brief geometric description of this subspace.
Solving $A\vec{x} = \vec{0}$ yields $\vec{x} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Use this to check that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the nullspace of A. From this, the dimension of the nullspace of A is 2, and this subspace is a plane through the origin in $\mathbb{R}^3$.
Note that all you need to know to figure out the dimension of the nullspace of A is the rank of A. If A is $m \times n$ and has rank r, then when solving $A\vec{x} = \vec{0}$, there are $n-r$ free variables. Each free variable will correspond to a basis element for the nullspace of A (all of which are linearly independent), and so $\dim(\operatorname{nullspace}(A)) = n-r$. This number is known as the nullity of A.