1. [14 points] Consider the differential equation $xy\dfrac{dy}{dx} = y^2 - 1$.

(a) Verify that $y = \sqrt{1 + Cx^2}$ is a solution of this differential equation if C is a constant.

If $y = \sqrt{1+Cx^2}$, then $\dfrac{dy}{dx} = \dfrac{Cx}{\sqrt{1+Cx^2}}$ and $xy\dfrac{dy}{dx} = x\sqrt{1+Cx^2}\,\dfrac{Cx}{\sqrt{1+Cx^2}} = Cx^2 = y^2 - 1$.

(b) Solve the initial value problem $xy\dfrac{dy}{dx} = y^2 - 1$, $y(1) = 3$.

Since $y = \sqrt{1+Cx^2}$ solves the D.E., $y(1) = 3$ yields $3 = \sqrt{1+C}$, so $C = 8$ and $y = \sqrt{1+8x^2}$ solves the I.V.P.

(c) Can you be certain that the initial value problem in part (b) has a unique solution? Explain.

Yes, the I.V.P. in part (b) has a unique solution. In standard form, the D.E. involved in this I.V.P. is $\dfrac{dy}{dx} = f(x,y)$, where $f(x,y) = \dfrac{y^2-1}{xy}$. Since $f(x,y)$ and $\dfrac{\partial f}{\partial y} = \dfrac{y^2+1}{xy^2}$ are continuous near the point $(1,3)$ (on a rectangle containing $(1,3)$), the I.V.P. has a unique solution by the existence and uniqueness theorem for first order differential equations.

2. [18 points] Solve each of the following differential equations.

(a) $\dfrac{dy}{dx} = (x-2)e^{-2y}$ (b) $\dfrac{dy}{dx} = \dfrac{9x^2 - 2x\sin y}{x^2\cos y + \sin y}$

(a) This D.E. is separable: $e^{2y}\,dy = (x-2)\,dx$. Integrating, we get $\frac{1}{2}e^{2y} = \frac{1}{2}x^2 - 2x + C$. This yields $y = \frac{1}{2}\ln(x^2 - 4x + A)$, where A is a constant.

(b) This D.E. is exact: In differential form $M(x,y)\,dx + N(x,y)\,dy = 0$, it can be written as $(2x\sin y - 9x^2)\,dx + (x^2\cos y + \sin y)\,dy = 0$. So $M = 2x\sin y - 9x^2$ and $N = x^2\cos y + \sin y$. Since $\dfrac{\partial M}{\partial y} = 2x\cos y = \dfrac{\partial N}{\partial x}$, the D. E. is exact. So there is a potential function $\phi = \phi(x,y)$ with $\dfrac{\partial \phi}{\partial x} = M = 2x\sin y - 9x^2$ and $\dfrac{\partial \phi}{\partial y} = N = x^2\cos y + \sin y$. Integrating the first of these equalities with respect to x yields $\phi = x^2\sin y - 3x^3 + h(y)$. Then, $\dfrac{\partial \phi}{\partial y} = x^2\cos y + h'(y) = x^2\cos y + \sin y$, so $h(y) = -\cos y$. So $\phi = x^2\sin y - 3x^3 - \cos y$ and $x^2\sin y - 3x^3 - \cos y = C$ solves the D.E.

3. [18 points] Consider the differential equation $y' - 2y = 3e^x y^3$.

(a) Explain why this differential equation is not linear.

Linear differential equations are linear functions of the dependent variable y and its derivatives. y^3 is not a linear function of y.

(b) Show that making the substitution $u = y^{-2}$ yields the differential equation $u' + 4u = -6e^x$.

If $u = y^{-2}$, then by the chain rule $u' = -2y^{-3}y'$, so $-\frac{1}{2}u' = y^{-3}y'$. Dividing the original D.E. by y^3 yields $y^{-3}y' - 2y^{-2} = 3e^x$. Substituting gives $-\frac{1}{2}u' - 2u = 3e^x$, i.e., $u' + 4u = -6e^x$.

(c) Solve the D.E. in part (b), and explain how this can be used to solve $y' - 2y = 3e^x y^3$.

The D.E. $u' + 4u = -6e^x$ is linear. The integrating factor is $I = e^{\int 4\,dx} = e^{4x}$. Multiplying by I yields $e^{4x}u' + 4e^{4x}u = -6e^{5x}$, i.e., $\dfrac{d}{dx}\left(e^{4x}u\right) = -6e^{5x}$. Integrating we get $e^{4x}u = -\frac{6}{5}e^{5x} + C$, so $u = -\frac{6}{5}e^x + Ce^{-4x}$. Since $u = y^{-2}$, substituting yields $y^{-2} = -\frac{6}{5}e^x + Ce^{-4x}$ which (implicitly) solves the original D.E.

4. [16 points] Consider the consistent system of linear equations: $\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 1 \\ 3x_1 & - & 6x_2 & + & x_3 & + & x_4 & = & 1 \end{array}$

Write the system in matrix form; find the free and bound variables; and find the solution set.

In matrix form, the system is $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 3 & -6 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 1 \\ 3 & -6 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R2 \to R2 - 3R1} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 1 \\ 0 & 0 & -2 & 4 & -2 \end{array}\right] \xrightarrow{R2 \to -\frac{1}{2}R2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

x_2, x_4 free, x_1, x_3 bound. Back substitution yields $x_1 = 2s - t$, $x_2 = s$, $x_3 = 1 + 2t$, $x_4 = t$.

5. [18 points] Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \\ 0 & 1 & 1 & 1 \\ 2 & 4 & 2 & 6 \end{bmatrix}$.

(a) Find a row echelon matrix that is row equivalent to A, and find the rank of A.

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \\ 0 & 1 & 1 & 1 \\ 2 & 4 & 2 & 6 \end{bmatrix} \xrightarrow[R4 \to R4 - 2R1]{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix} \xrightarrow[R4 \to R4 - 2R2]{R3 \to R3 - R2} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{rank}\, A = 2$$

Answer the following questions based on your results in part (a). Include brief explanations.

(b) Is the matrix A invertible?
No, since $\text{rank}\, A = 2 < 4$, A is not invertible.

(c) If $\vec{b}$ is a 4×1 vector, is the system of linear equations $A\vec{x} = \vec{b}$ necessarily consistent?
No, $A\vec{x} = \vec{b}$ is consistent for any $\vec{b}$ if and only if A is invertible if and only if $\text{rank}\, A = 4$. But $\text{rank}\, A = 2 < 4$. Can you find a vector $\vec{b}$ for which the system $A\vec{x} = \vec{b}$ has no solution?

(d) How many solutions does the homogeneous system of linear equations $A\,\vec{x} = \vec{0}$ have?
Infinitely many, since $\text{rank}\, A = 2$ the homogeneous system will have $4 - 2 = 2$ free variables.

6. [16 points] Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ -1 \\ 8 \end{bmatrix}$. The inverse of A is $A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 1 \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to calculate the inverse of the matrix A by hand.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3 \to R3 - R1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R2 \to R2 - R3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R1 \to R1 + R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & -2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R2 \to -\frac{1}{2}R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

(b) Use the inverse of the matrix A to solve the system of linear equations $A\,\vec{x} = \vec{b}$.

$$A\,\vec{x} = \vec{b} \implies A^{-1}A\vec{x} = A^{-1}\vec{b} \implies I\vec{x} = A^{1}\vec{b} \implies \vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$