1. [13 points] In part (a), determine if the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defines a linear transformation.

(a) $T(x_1, x_2) = (x_1 + 2x_2, 3 + 4x_1)$
Since, for instance, $T(\vec{x}+\vec{y}) = T(x_1+y_1, x_2+y_2) = (x_1+y_1+2x_2+2y_2, 3+4x_1+4y_1)$ and $T(\vec{x})+T(\vec{y}) = T(x_1,x_2)+T(y_1,y_2) = (x_1+y_1+2x_2+2y_2, 6+4x_1+4y_1)$, we have $T(\vec{x}+\vec{y}) \neq T(\vec{x})+T(\vec{y})$, and T is not a linear transformation.

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $T(1,0) = (1,3)$ and $T(1,1) = (3,3)$, find $T(-1,3)$.
Since $(-1,3) = 3(1,1) - 4(1,0)$, $T(-1,3) = T(3(1,1) - 4(1,0)) = 3T(1,1) - 4T(1,0) = (5,-3)$.

2. [13 points] Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -3x_1 + 3x_2 - 6x_3 \end{bmatrix}$.

(a) Find the matrix of T.

The matrix of T is $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 3 & -6 \end{bmatrix}$.

(b) Find the dimensions of the kernel and range of T.
Since the matrix A of T has rank 1, the dimension of the kernel of T is 2 (the kernel of T is the nullspace of A, and there are 2 free variables). Then, for instance by the rank-nullity theorem, the dimension of the range of T is 1, since $\dim \mathrm{Ker}(T) + \dim \mathrm{Rng}(T) = \dim \mathbb{R}^3 = 3$. (Alternatively, the range of T is the column space of A, i.e., the subspace of $\mathbb{R}^2$ spanned by the columns of A. From part (a), it is clear that this subspace is spanned by, e.g., the first column of A, so has dimension 1. See problem 5 on Exam 2.)

3. [13 points] Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 4 & 6 \\ 1 & 0 & 3 \end{bmatrix}$.

Expanding along column 2, the characteristic polynomial of A is:

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -1 \\ 2 & 4-\lambda & 6 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(\lambda^2 - 6\lambda + 10)$$

Using the quadratic formula, the eigenvalues of A are $\lambda_1 = 4$, $\lambda_2 = 3+i$, $\lambda_3 = 3-i$.

4. [13 points] Suppose that A is a 2×2 matrix with eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ associated to the eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 6$ respectively. Find the matrix A.

We are given that $A\vec{v}_1 = \lambda_1\vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$. Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A\vec{v}_1 = \lambda_1\vec{v}_1$ reads $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $\begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$. Similarly, from $A\vec{v}_2 = \lambda_2\vec{v}_2$, we get $\begin{bmatrix} a+3b \\ c+3d \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$.
This yields a system of 4 linear equations in 4 unknowns: $a+2b = 4$, $a+3b = 6$, $c+2d = 8$, $c+3d = 18$. Solving yields $A = \begin{bmatrix} 0 & 2 \\ -12 & 10 \end{bmatrix}$.

Alternatively, the matrix A is diagonalizable (why?), $S^{-1}AS = D$, where $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$. Since $S^{-1}AS = D$, we have $A = SDS^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$.

5. [14 points] The matrix $A = \begin{bmatrix} 1 & -3 & 1 \\ -1 & -1 & 1 \\ -1 & -3 & 3 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = -(\lambda - 2)^2(\lambda + 1)$.

Determine if A is defective or diagonalizable. If A is diagonalizable, find a diagonal matrix D and an invertible matrix S so that $S^{-1}AS = D$. (You do not have to find S^{-1}.)

A has eigenvalues $\lambda_1 = \lambda_2 = 2$ (algebraic multiplicity 2) and $\lambda_3 = -1$ (algebraic multiplicity 1).

$A - 2I = \begin{bmatrix} -1 & -3 & 1 \\ -1 & -3 & 1 \\ -1 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Solving $(A - 2I)\vec{v} = \vec{0}$ yields two linearly independent eigenvectors, e.g., $\vec{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

So the eigenvalue 2 has geometric multiplicity 2.

$A + I = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 0 & 1 \\ -1 & -3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Solving $(A + I)\vec{v} = \vec{0}$ yields $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

So the eigenvalue -1 has geometric multiplicity 1.

The calculations above show that A has a complete set of eigenvectors, so A is diagonalizable,

$S^{-1}AS = D$, where $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

6. [21 points] Give real-valued solutions.

(a) Solve the initial value problem $y'' - 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 5$.

The auxiliary polynomial is $r^2 - 6r + 9 = (r - 3)^2$, so the DE has general solution $y = c_1e^{3x} + c_2xe^{3x}$. Note that $y' = 3c_1e^{3x} + c_2e^{3x} + 3c_2xe^{3x}$. The initial conditions $y(0) = 1$ and $y'(0) = 5$ yield $c_1 = 1$ and $3c_1 + c_2 = 5$, so $c_2 = 2$. Consequently, $y = e^{3x} + 2xe^{3x}$ solves the IVP.

(b) Find the general solution of the differential equation $(D - 4)(D^2 - 6D + 10)y = 0$.

Since the auxiliary polynomial $(r - 4)(r^2 - 6r + 10)$ has roots 4, $3 + i$, $3 - i$, the DE has general solution $y = c_1e^{4x} + c_2e^{3x}\cos(x) + c_3e^{3x}\sin(x)$.

(c) Verify that $y_p = e^{4x} - 1$ is a particular solution of the differential equation $y'' - 9y = 7e^{4x} - 9$, and find the general solution of this differential equation.

As some of you noted, there was a typo on the exam. The particular solution should have been $y_p = e^{4x}+1$. Here is a solution with this correction included:

If $y_p = e^{4x} + 1$, then $y_p' = 4e^{4x}$, and $y_p'' = 16e^{4x}$, so $y_p'' - 9y_p = 16e^{4x} - 9e^{4x} - 9 = 7e^{4x} - 9$.

The auxiliary polynomial is $r^2 - 9 = (r - 3)(r + 3)$, so $y_c = c_1e^{3x} + c_2e^{-3x}$ is the complementary solution. Thus, $y = y_c + y_p = c_1e^{3x} + c_2e^{-3x} + e^{4x} - 1$ is the general solution of the given non-homogeneous DE.

In light of the typo, I gave full credit on this part for any reasonable response.

7. [13 points] Consider the differential equation $x^2y'' - xy' - 3y = 0$, $x > 0$.

(a) Find the two values of r for which the function $y(x) = x^r$ is a solution of this differential equation.

If $y = x^r$, then $y' = rx^{r-1}$, and $y'' = r(r - 1)x^{r-2}$, so $x^2y'' - xy' - 3y = r(r - 1)x^r - rx^r - 3x^r = x^r(r^2 - 2r - 3) = x^r(r - 3)(r + 1)$. Since $x > 0$, this is equal to zero if $r = 3$ or $r = -1$. So $y_1 = x^3$ and $y_2 = x^{-1}$ are solutions of the DE.

(b) If r_1 and r_2 are the two values of r from part (a), does the set of functions $\{x^{r_1}, x^{r_2}\}$ form a basis for the solution space of this differential equation on the interval $(0, \infty)$? Explain.

Since $W[x^3, x^{-1}] = \begin{vmatrix} x^3 & x^{-1} \\ 3x^2 & -x^{-2} \end{vmatrix} = -4x \neq 0$ on the interval $(0, \infty)$, the set $\{x^3, x^{-1}\}$ is linearly independent on $(0, \infty)$. Since x^3 and x^{-1} are linearly independent solutions of the given second order homogeneous linear DE, the set $\{x^3, x^{-1}\}$ is a basis for the solution space of this DE.

Note that this part can be done without completing part (a). If x^{r_1} and x^{r_2} are solutions of the given DE with $r_1 \neq r_2$, then since $W[x^{r_1}, x^{r_2}] = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1x^{r_1-1} & r_2x^{r_2-1} \end{vmatrix} = (r_2 - r_1)x^{r_1+r_2-1} \neq 0$ on $(0, \infty)$, the set $\{x^{r_1}, x^{r_2}\}$ is linearly independent, so forms a basis for the solution space as above.