Exam 3 will take place on Monday, April 15. It will cover material we've discussed from Chapters 5 and 6 in the text (specifically, §§5.1, 5.3, 5.6, 5.7, 5.8, 6.1, 6.2). Some remarks concerning material on the exam are included below.

Books, notes, calculators, computers, smart phones, etc. may **not** be used on the exam.

If you have questions regarding this material, be ready to ask them in class this week. You may also make use of my office hours, and the free tutoring available in 141 Middleton Library (hours: M–Th 10:30–7:00, F 10:30–3:00). I don't know if people capable of tutoring for MATH 2090 are available.

A few review problems are included below. This is **not** a comprehensive list. Additional problems may be found in the Exercises of the sections we've covered, and in the WeBWorK assignments. For review/practice with primarily computational problems (and/or to improve your homework grade), I have reopened all the relevant WeBWorK assignments. They will remain open through Monday, April 15. For more conceptual aspects, refer to appropriate problems assigned from the text. You may also find the True-False Reviews at the end of each section, and the Chapter Reviews useful in this regard.

Chapter 5. Linear Transformations and the Eigenvalue/Eigenvector Problem

Know what a linear transformation is, and be able to determine if a given mapping between vector spaces is one [§5.1]. You should also know what the kernel and range of a linear transformation are, and be able to find bases for them [§5.3]. This problem – find the kernel of a linear transformation – sums up much of the course. Think about why. Be able to find the eigenvalues, eigenvectors, eigenspaces of a square matrix A, to determine if A is defective [§5.6, §5.7], and to determine if A is diagonalizable [§5.8]. Note that you must be able to compute determinants, solve systems, etc. correctly (and efficiently) to find the eigenvalues/eigenvectors of a matrix (in a timely manner).

Chapter 6. Linear Differential Equations of Order n

Be prepared to work with general and particular solutions of homogeneous and non-homogeneous linear ODE's (as discussed in §6.1). Be able to solve constant coefficient linear ODE's and IVP's, by finding the roots of the auxiliary polynomial for homogeneous (or complementary) solutions [§6.2]. You should be able to produce real-valued solutions (using Euler's formula if need be, as in §6.2). Be prepared to work with the underlying theory discussed in §6.1 (Theorem 6.1.3, Theorem 6.1.7, the role/utility of the Wronskian, etc.).

Review Problems

1. (a) Consider the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_3, k)$, where k is a scalar. Find all values of k for which T is a linear transformation. Find the matrix of T for these values of k.

(b) If $T : \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation and $T(1, 0, 0) = 1$, $T(1, 1, 0) = 3$, $T(1, 1, 1) = 5$, find $T(2, 4, 6)$.

2. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 3 & 1 \\ -4 & 0 & 2 \end{bmatrix}$.

(a) Find a basis for Ker(T), the kernel of the linear transformation T. What is the dimension of Ker(T)?

(b) Find a basis for Rng(T), the range of the linear transformation T. What is the dimension of Rng(T)?

3. For each of the matrices A below, determine if A is defective or diagonalizable. If A is diagonalizable, find a diagonal matrix D and an invertible matrix S so that $S^{-1}AS = D$.

(a) $A = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 4 & -5 \\ 1 & 2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 7 & -4 \\ 4 & -1 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$ (e) $A = \begin{bmatrix} 2 & -3 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix}$

4. Suppose λ is an eigenvalue of the matrix A, and that $\vec{v}$ is a corresponding eigenvector.

(a) Show that λ is an eigenvalue of the matrix A^T.

(b) Show that λ^2 is an eigenvalue of the matrix A^2, and that $\vec{v}$ is a corresponding eigenvector.

(c) If A is invertible, show that $1/\lambda$ is an eigenvalue of A^{-1}, and that $\vec{v}$ is a corresponding eigenvector.

5. The differential equation $x^3y''' + x^2y'' - 2xy' + 2y = 0$, $x > 0$ has three solutions of the form $y(x) = x^r$. Find these solutions, show that they are linearly independent on the interval $(0, \infty)$, and find the general solution of this differential equation. Then solve the initial value problem involving this differential equation, together with the initial conditions $y(1) = 3$, $y'(1) = 2$, $y''(1) = 0$.

6. For each differential equation below, find the general solution. Also solve the initial value problem in part (a). In part (d), first verify that $y_p = \frac{8}{5}e^{3x}$ is a particular solution. In part (e), first find a particular solution of the form $y_p = A\cos(x) + B\sin(x)$, where A and B are constants.

(a) $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 6$ (b) $y'' + 10y' + 25y = 0$ (c) $y'' - 2y' + 5y = 0$

(d) $y'' - 4y = 8e^{3x}$ (e) $y'' - 4y = 4\cos(x)$ (f) $D^2(D-9)^3(D^2+9)y = 0$

Answers to the Review Problems

1. (a) T is a linear transformation only when $k = 0$. If $k = 0$, the matrix of T is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $T(2, 4, 6) = T(-2(1, 0, 0) - 2(1, 1, 0) + 6(1, 1, 1)) = -2T(1, 0, 0) - 2T(1, 1, 0) + 6T(1, 1, 1) = 22$

2.

(a) For instance, $\left\{ \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix} \right\}$ is a basis for $\text{Ker}(T)$. $\dim \text{Ker}(T) = 1$.

(b) For instance, $\left\{ \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right\}$ is a basis for $\text{Rng}(T)$. $\dim \text{Rng}(T) = 2$

3.

(a) diagonalizable: $D = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$, $S = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$

(b) diagonalizable: $D = \begin{bmatrix} 3+2i & 0 \\ 0 & 3-2i \end{bmatrix}$, $S = \begin{bmatrix} 1+2i & 1-2i \\ 1 & 1 \end{bmatrix}$

(c) defective: $\lambda = 3$ has algebraic multiplicity 2/geometric multiplicity 1 - the associated eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(d) diagonalizable: $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $S = \begin{bmatrix} -3 & 1 & 1 \\ 9 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$

(e) defective: $\lambda = 3$ has algebraic multiplicity 2/geometric multiplicity 1 - the associated eigenspace is span $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

4. λ is an eigenvalue of the matrix A and that $\vec{v}$ is a corresponding eigenvector: $A\vec{v} = \lambda\vec{v}$ (and $\vec{v} \neq \vec{0}$)

(a) Since $\det(A^T - \lambda I) = \det(A - \lambda I)^T = \det(A - \lambda I)$, the characteristic polynomials of A and A^T are the same. So A and A^T have the same eigenvalues.

(b) Since $A\vec{v} = \lambda\vec{v}$, $A^2\vec{v} = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda^2\vec{v}$

(c) $\lambda \neq 0$ since A is invertible. Since $A\vec{v} = \lambda\vec{v}$, $\vec{v} = A^{-1}A\vec{v} = A^{-1}(\lambda\vec{v}) = \lambda A^{-1}\vec{v}$. So $(1/\lambda)\vec{v} = A^{-1}\vec{v}$.

5. The general solution of the D.E. is $y = c_1x + c_2x^2 + c_3x^{-1}$. The solution of the I.V.P. is $y = x + x^2 + x^{-1}$.

6.

(a) $y = c_1e^{2x} + c_2e^{-2x}$ solves the D.E., $y = 2e^{2x} - e^{-2x}$ solves the I.V.P.

(b) $y = c_1e^{-5x} + c_2xe^{-5x}$

(c) $y = c_1e^x\cos(2x) + c_2e^x\sin(2x)$

(d) $y = c_1e^{2x} + c_2e^{-2x} + \frac{8}{5}e^{3x}$

(e) $y = c_1e^{2x} + c_2e^{-2x} - \frac{4}{5}\cos(x)$

(f) $y = c_1 + c_2x + c_3e^{9x} + c_4xe^{9x} + c_5x^2e^{9x} + c_6\cos(3x) + c_7\sin(3x)$