Exam 2 will take place on Thursday, March 14. It will cover material we've discussed from Chapters 3 and 4 in the text. Some remarks concerning the material on the exam are included below.

Books, notes, calculators, computers, smart phones, etc. may **not** be used on the exam.

If you have questions regarding this material, be ready to ask them in class next week. You may also make use of my office hours, and the free tutoring available in 141 Middleton Library (hours: M–Th 10:30–7:00, F 10:30–3:00). I don't know if people capable of tutoring for MATH 2090 are available.

A few review problems are included below. This is **not** a comprehensive list. Additional problems may be found in the Exercises of the sections we've covered, and in the WeBWorK assignments. For review/practice with primarily computational problems (and/or to improve your homework grade), I have reopened all the relevant WeBWorK assignments. They will remain open through Thursday, March 14. For more conceptual aspects, refer to appropriate problems assigned from the text. You may also find the True-False Reviews at the end of each section, and the Chapter Reviews useful in this regard.

Chapter 3. Determinants

You should be able to compute determinants using various properties, or cofactor expansion, or some combination. These are summarized in §3.4 and §3.5, and are discussed in more detail in §3.2 and §3.3. Understand the implications of your calculations. For instance, what is the relation between $\det A$ and the invertibility of A? between $\det A$ and systems of linear equations involving A? between $\det A$ and the linear independence or dependence of the columns of A? etc.

Chapter 4. Vector Spaces

I won't ask you to work with the definition of a vector space [§4.2], but you should be aware of what one is (informally, a set of "vectors" which we can add, and multiply by scalars). I may (read as "will') ask you if a subset of a "known" vector space (e.g. $\mathbb{R}^n$, P_n – polynomials of at most n, $C^k(I)$ – functions with k continuous derivatives on the interval I, $M_{m,n}(\mathbb{R})$ – $m \times n$ matrices with real entries, ...) is a subspace [§4.3]. You should be able to work with various vector space notions: linear combinations, span [§4.4]; linear dependence and independence [§4.5] (and independence of functions and the Wronskian); basis and dimension [§4.6 and the beginning of §4.7]. Questions regarding these notions can often be reduced to questions about systems of linear equations [see Chapter 2]. You should be prepared to make these reductions (quickly), deal with the resulting linear system (quickly), and interpret your results in terms of the aforementioned vector space notions.

Review Problems

1. Consider the matrix $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & 3 & -2 & 4 \end{bmatrix}$.
   (a) Compute the determinant of A. Does the matrix A have an inverse? Explain.
   (b) If B is a 4×4 matrix with $\det(B) = -7$, what is the determinant of the product AB? Explain.
   (c) Can a system $A\vec{x} = \vec{b}$ involving this matrix A, and any constant vector $\vec{b}$, be inconsistent? Explain.

2. Consider the vectors $\vec{v}_1 = (1, 3, 4)$, $\vec{v}_2 = (0, 1, 1)$, $\vec{v}_3 = (-1, 0, -1)$, $\vec{v}_4 = (5, 3, -2)$ in $\mathbb{R}^3$.
   (a) Does the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ span $\mathbb{R}^3$? (b) Is this set of vectors is linearly independent?
   (c) Does this set of vectors form a basis for $\mathbb{R}^3$? Explain.

3. Recall that the trace of a square matrix is the sum of the diagonal entries. Let S be the set of all 2×2 matrices with real entries and zero trace: $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;:\; a, b, c, d \text{ real, and } a + d = 0 \right\}$.
   (a) Show that S is a subspace of $M_2(\mathbb{R})$, the vector space of all 2×2 matrices with real entries.
   (b) Verify that $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for S.
   (c) Find the components of the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ with respect to the basis in part (b).
   In other words, express the matrix A as a linear combination of the matrices given in part (b).

4. Consider the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 3 & 1 \\ -4 & 0 & 2 \end{bmatrix}$.
   (a) Find a basis for the nullspace of A. What is the dimension of the nullspace of A?
   (b) Find a basis for the subspace of $\mathbb{R}^3$ spanned by the columns of A. What is the dimension of this subspace?
   (c) Do the rows of the matrix A form a basis for $\mathbb{R}^3$? Explain.

Answers to the Review Problems

1. (a) $\det A = -7$. Since $\det A \neq 0$, A **is** invertible.
   (b) $\det AB = \det A \det B = 49$.
   (c) Since A is invertible, for any $\vec{b}$, the system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$. In particular, $A\vec{x} = \vec{b}$ is consistent for any $\vec{b}$.

   The columns of A form a basis for $\mathbb{R}^4$. Which aspect of "basis" does part (c) above refer to? How do you explain why the other aspect of "basis" holds?

2. (a) This set of vectors **does** span $\mathbb{R}^3$.
   (b) This set of vectors **is not** linearly independent.
   Any set of 4 vectors in $\mathbb{R}^3$ must be linearly dependent. Explain why.
   (c) Since this set of vectors is linearly dependent, it does not form a basis for $\mathbb{R}^3$.

   You should be able to find a subset of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ which **is** a basis for $\mathbb{R}^3$. How would you accomplish this?

3. (a) We must show that S is closed under vector addition and scalar multiplication.
   Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be elements of S.
   Then the trace of A is 0, i.e., $a + d = 0$, and the trace of B is 0, i.e., $p + s = 0$.
   The trace of $A + B = \begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix}$ is $(a+p) + (d+s) = (a+d) + (p+s) = 0 + 0 = 0$. So $A + B$ is in S, and S is closed under vector addition.
   If k is a scalar and A (above) is in S, then the trace of $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$ is $ka + kd = k(a+d) = k0 = 0$.
   So kA is in S, and S is closed under scalar multiplication.

   (b) We must show that the given set is a linearly independent spanning set. Let $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be an arbitrary element of S. Then, as above, $p + s = 0$, i.e., $s = -p$. Consider the matrix equation

   $$c_1 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p & q \\ r & -p \end{bmatrix}.$$

   Check that this equality holds if and only if $c_1 - c_3 = p$, $c_1 + c_2 + c_3 = q$, $c_1 + c_2 = r$, $-c_1 + c_3 = -p$. The latter is a system of 4 linear equations in the 3 unknowns c_1, c_2, c_3. Use Gaussian elimination to check that this system has a unique solution for any p, q, r. It follows from this that the given set of matrices spans S. Also, taking $p = q = r = 0$, i.e., $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the above system has only the trivial solution $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. Consequently, the given set of matrices is also linearly independent.

   (c) $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = (2)\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (1)\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

4. Consider the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 3 & 1 \\ -4 & 0 & 2 \end{bmatrix}$.

   (a) For instance, $\left\{ \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix} \right\}$ is a basis for the nullspace of A.
   The dimension of the nullspace of A is 1.

   (b) For instance, $\left\{ \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right\}$ is a basis for the subspace of $\mathbb{R}^3$ spanned by the columns of A.
   The dimension of this subspace is 2.
   This subspace is called the column space of A.

   (c) The rows of A **do not** form a basis for $\mathbb{R}^3$. Since $A\vec{x} = \vec{0}$ has non-trivial solutions (see (a)), A is not invertible, the columns of A are linearly dependent, $\det A = 0$, etc. Since $\det A = \det A^\top$, we also have $\det A^\top = 0$. So $A^\top \vec{y} = \vec{0}$ has non-trivial solutions, and the columns of $A^\top$ (i.e., the rows of A) are linearly dependent. Since the rows of A are linearly dependent, they cannot form a basis.
   The subspace spanned by the rows of A is called the row space of A. For this matrix A, you can check that the first two rows of A form a basis for the row space, so the dimension of the row space is 2. This is not an accident. It is always the case that the $\dim(\text{column space of } A) = \dim(\text{row space of } A) = \operatorname{rank} A$.