MATH 2090-3 Exam 1 Review Spring 2013

Exam 1 will take place on Thursday, February 7. It will cover material we've discussed from Chapters 1 and 2 in the text. Some remarks concerning the material on the exam are included below.

Books, notes, calculators, computers, smart phones, etc. may **not** be used on the exam.

If you have questions regarding this material, be ready to ask them in class next week. You may also make use of my office hours, and the free tutoring available in 141 Middleton Library (hours: M–Th 10:30–7:00, F 10:30–3:00). I don't know if people capable of tutoring for MATH 2090 are available.

Some review problems are included on the next page. This is **not** a comprehensive list. Additional problems may be found in the Exercises of the sections we've covered, and in the WeBWorK assignments. For review/practice with primarily computational problems (and/or to improve your homework grade), I have reopened all the relevant WeBWorK assignments. They will remain open through Thursday, February 7. For more conceptual aspects, refer to appropriate problems assigned from the text. You may also find the True-False Reviews at the end of each section, and the Chapter Reviews useful in this regard.

Chapter 1. First-Order Differential Equations

Be comfortable with the basic notions and terminology discussed in §1.2 and elsewhere. The main classes of first-order DEs we discussed are separable [§1.4 - separate the variables, then integrate], linear [§1.6 - integrating factor], and exact [§1.9 - test for exactness using partial derivatives, if exact, find potential function by integration]. You should be prepared to (quickly) recognize and solve DEs of these forms. Some other classes, e.g., homogeneous, Bernoulli, may be related to these by appropriate substitutions, see §1.8. I will also expect you to know what the existence and uniqueness theorem (Theorem 1.3.2 on p. 22) for first-order IVPs says, and to be able to work with it.

In an exam setting, I will attempt to avoid lengthy/delicate integration problems. But I will expect you to be able to (quickly) carry out standard integration techniques, such as substitution, parts, partial fractions, etc.

Chapter 2. Matrices and Systems of Linear Equations

Be comfortable with the basic notions, terminology, matrix algebra, etc. discussed at the beginning of the chapter. You should be proficient at matrix operations: can add/subtract, multiply matrices, take transposes, know what the identity matrix is, translate a system of linear equations into matrix form... You should be able to reduce matrices to row echelon form, find the rank, find solutions sets of systems of linear equations using row operations/Gaussian elimination, find the inverse of a square matrix (or determine if no inverse exists), etc. You should also be prepared to interpret your results and understand their implications. For instance, if you know the rank of a matrix A, what can you say about systems of equations involving A? If A is square, what are the relations between rank A, systems involving A, the invertibility of A, etc.?

Review Problems

1. Solve the initial value problems $y' = 4x\sqrt{y-1},\ y(2) = 1$ and $y' = y^2 \sin x,\ y(0) = 1$.
   Can you be certain if either of these initial value problems has a unique solution?

2. Find the general solution of each of the following differential equations.

   (a) $xy' - 2y = 2x^2 \ln x$ (b) $x + xy^2 + e^{x^2} y \dfrac{dy}{dx} = 0$ (c) $\dfrac{dy}{dx} = \dfrac{\sin y + y\cos x + 1}{1 - x\cos y - \sin x}$

   (d) $xy' - y = 4xy^2$ (substitute $u = y^{-1}$) (e) $\dfrac{dy}{dx} = \dfrac{x^2}{x^2 - y^2} + \dfrac{y}{x}$ (substitute $V = y/x$)

3. Newton's law of cooling states that the rate of change of the temperature of an object with respect to time t is proportional to difference between the temperature, $W(t)$, of the object at time t and the temperature, R, of the surrounding medium.
   Suppose that the temperature of the water in my bottle is 45°F at the beginning of class, and is 50°F ten minutes later. Assume that room temperature $R = 72$°F is constant. To determine the temperature of my water at the end of class using Newton's law of cooling, you would have to solve a certain initial value problem. Write this initial value problem down. Then solve it.

4. Do problems #1–7 odd on page 185 of the text.

5. Consider the system of linear equations
$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & + & x_4 & = & 6 \\ 2x_1 & - & x_2 & + & 3x_3 & - & x_4 & = & 3 \\ -x_1 & + & x_2 & - & 2x_3 & & & = & -3 \end{array}$$
   Write this system in matrix form, find the rank of the coefficient matrix, and find the solution set of the system.

6. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ and the constant vector $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Use Gauss-Jordan elimination to find the inverse of A.

   (b) Use your answer from part (a) to solve the system of equations $A\vec{x} = \vec{b}$.

   (c) Is the transpose of A invertible? Explain.

   (d) Can a system $A\vec{x} = \vec{b}$ involving this matrix A, and any constant vector $\vec{b}$, be inconsistent? Explain.

7. Consider the homogeneous system of linear equations
$$\begin{array}{rcrcrcl} (1-\lambda)x_1 & + & 2x_2 & + & 6x_3 & = & 0 \\ & & (2-\lambda)x_2 & + & 3x_3 & = & 0 \\ & & x_2 & + & (4-\lambda)x_3 & = & 0 \end{array}$$

   (a) Find all values of λ for which this system has infinitely many solutions.

   (b) Find the solution set of this homogeneous system when $\lambda = 1$.

Answers to the Review Problems

1. Using separation of variables, a solution of $y' = 4x\sqrt{y-1}$, $y(2) = 1$ is $y = 1 + (x^2-4)^2$. This solution is **not** unique. For instance, the function $y = 1$ is also a solution. The existence and uniqueness theorem does not apply to this IVP.

   Using separation of variables, a solution of $y' = y^2 \sin x$, $y(0) = 1$ is $y = \sec x$. The existence and uniqueness theorem does apply to this IVP, and this solution is unique.

2. (a) $y = x^2(\ln x)^2 + Cx^2$ (b) $\ln(1+y^2) = e^{-x^2} + C$ (c) $x \sin y + y \sin x + x - y = C$

   (d) $y = \dfrac{x}{C - 2x^2}$ (e) $\dfrac{y}{x} - \dfrac{y^3}{3x^3} = \ln x + C$

3. $\dfrac{dW}{dt} = k(W-72)$, $W(0) = 45$, also $W(10) = 50$ $\quad W(t) = 72 - 27\left(\dfrac{22}{27}\right)^{t/10}$

4. See page 769 of the text.

5. In matrix form, the system is $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & -1 & 3 & -1 \\ -1 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}$.

   The rank of the coefficient matrix is two.
   The solution set of the system is $\{(-s+t, s+t-3, s, t) : s, t \text{ real}\}$.

6. (a) $A^{-1} = \begin{bmatrix} 5 & -1 & -2 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (b) $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 12 \\ 5 \\ -2 \end{bmatrix}$ (c) Yes, $(A^T)^{-1} = (A^{-1})^T$.

   (d) Since A is invertible, any system with coefficient matrix A has a unique solution (so is consistent).

7. (a) The system has infinitely many solutions if $\lambda = 1$ or $\lambda = 5$.
   (b) When $\lambda = 1$, the solution set is $\{(s, -3t, t) : s, t \text{ real}\}$