Math 2090-3 **Final Exam Information** **Spring 2013**

Final Exam: Saturday, May 11, 10:00 a.m. – 12:00 noon, in 220 Stubbs (our usual classroom). Books, notes, calculators, computers, smart phones, etc. may **not** be used.

Office Hours: I anticipate being available in my office (372 Lockett) Monday through Friday of finals week from 8:30 to 9:30 a.m. each morning.
You may also contact me by email at `cohen@math.lsu.edu` at any time.

Materials: All homework assignments, review sheets, and answers to the problems on the exams are posted. Follow the links from `http://www.math.lsu.edu/~cohen/courses/FALL01/M2090syl.html`. My answers to the exam problems and the problems on the review sheets are posted on the exams page, as are the review sheets themselves. The "new" topics you are responsible for on the final include §6.7 (variation-of-parameters to find particular solutions of non-homogeneous ODE), the material we've discussed on systems of DE (§§7.1–7.4, 7.6), and the material we've discussed regarding the Laplace transform (§§8.1–8.5). Remarks, and a few representative examples, concerning these topics are included below. Refer to previously posted review sheets for remarks concerning topics covered on the in-class exams. Suitable review problems may be found in the Exercises of the sections we've covered, and in the WeBWorK assignments. For review/practice with primarily computational problems (and/or to improve your homework grade), all WeBWorK assignments have been reopened, and will remain open through Saturday, May 11. For more conceptual aspects, refer to appropriate problems assigned from the text. You may also find the True-False Reviews at the end of each section, and the Chapter Reviews useful. The final will be cumulative, and will address topics we've covered this semester in a proportionate manner.

Chapter 6. Linear Differential Equations of Order *n* Be prepared to work with general and particular solutions of homogeneous and non-homogeneous linear ODE's (as discussed in §6.1). Be able to solve constant coefficient linear ODE's and IVP's, by finding the roots of the auxiliary polynomial for homogeneous (or complementary) solutions [§6.2], and by using variation-of-parameters to find particular solutions of non-homogeneous equations [§6.7]. You should be able to produce real-valued solutions (using Euler's formula if need be, as in §6.2). Be prepared to work with the underlying theory discussed in §6.1 (Theorem 6.1.3, Theorem 6.1.7, the role/utility of the Wronskian, etc.). I will not explicitly ask you about existence/uniqueness theorems on the final (in this context, or in the context of a 1st order DE or a system), but you should, for instance, know what a solution of a DE is, understand that a homogeneous linear nth order ODE has n linearly independent solutions, be able to determine if solutions are independent…

Chapter 7. Systems of Differential Equations Be able to solve homogeneous constant coefficient systems, with diagonalizable coefficient matrix [§7.4]. As in Ch. 6, if you're given a real system $\vec{x}\,' = A\,\vec{x}$, you should be able to generate real-valued solutions (see §7.4, e.g., Example 7.4.6). Be aware that the method of variation-of-parameters may be applied in the context of systems [§7.6]. Also, be comfortable with the relevant terminology (e.g. solution, fundamental solution set, fundamental matrix, etc.), and be aware of the underlying theory [§§7.1–7.3]. For instance, under what conditions is a set of solutions a fundamental set of solutions? What is the Wronskian in this context? What is it's use?

Note that I can ask you to find determinants, solve the eigenvalue problem, etc. implicitly: "Find the general solution of the system $\vec{x}\,' = A\,\vec{x}$." For example, referring to problem #3 on the Exam 3 Review, for which of the matrices given there can you find a fundamental set of solutions for the system $\vec{x}\,' = A\,\vec{x}$?

Chapter 8. The Laplace Transform ... Be able to solve constant coefficient linear initial value problems using the Laplace transform [§§8.4–8.5]. For this, among other things, you will need to be able to carry out partial fractions decompositions (reviewed in Appendix B). While one of the strengths of the Laplace transform is its ability to handle IVPs with discontinuous forcing function [§8.7], I will not ask you about this on the final. I will provide a table of Laplace transforms and the first shifting theorem. I will not provide the relation between the Laplace transform of a function and its derivative – this you should commit to memory. You should also know the definition and properties of the Laplace transform [§§8.1–8.2], be able to calculate Laplace transforms using the definition, be able to calculate Laplace transforms and inverse Laplace transforms of functions using a table, etc..

Examples

1. Use the definition of the Laplace transform to find the Laplace transform of the function $f(t) = e^{-3t}$.

$$L[e^{-3t}] = \int_0^\infty e^{-st}e^{-3t}\,dt = \lim_{N\to\infty}\int_0^N e^{-(s+3)t}\,dt = \lim_{N\to\infty} \frac{-e^{-(s+3)t}}{s+3}\bigg|_0^N = \lim_{N\to\infty}\left[\frac{1}{s+3} - \frac{e^{-(s+3)N}}{s+3}\right] = \frac{1}{s+3}$$

provided $s > -3$

2. Solve the initial value problem $y'' + y' - 2y = 3e^{-2t}$, $y(0) = 3$, $y'(0) = -1$.
Here are three solutions, illustrating the methods from Chapters 6, 7, and 8.

(i) Using variation-of-parameters, etc.: The complementary homogeneous ODE $y'' + y' - 2y = 0$ has auxiliary polynomial $r^2 + r - 2 = (r+2)(r-1)$, so the complementary solution is $y_c = c_1e^{-2t} + c_2e^t$. Using variation-of-parameters, we look for a particular solution of the original non-homogeneous ODE of the form $y = u_1(t)e^{-2t} + u_2(t)e^t$, where the functions $u_1 = u_1(t)$ and $u_2 = u_2(t)$ satisfy $e^{-2t}u_1' + e^tu_2' = 0$ and $-2e^{-2t}u_1' + e^tu_2' = 3e^{-2t}$ (see pp. 503–504 in §6.7 for details). Solving for u_1' and u_2' yields $u_1' = -1$ and $u_2' = e^{-3t}$. So, for instance, $u_1 = -t$, $u_2 = -\frac{1}{3}e^{-3t}$, and $y = u_1e^{-2t} + u_2e^t = -te^{-2t} - \frac{1}{3}e^{-2t}$ is a solution of the non-homogeneous ODE. Note that $-\frac{1}{3}e^{-2t}$ is already a homogeneous solution. So we can take $y_p = -te^{-2t}$ as a particular solution, and the general solution of the non-homogeneous ODE is $y = y_c + y_p = c_1e^{-2t} + c_2e^t - te^{-2t}$. The initial conditions $y(0) = 3$, $y'(0) = 1$ give the equations $c_1 + c_2 = 3$, $2c_1 - c_2 = 0$, which yield $c_1 = 1$, $c_2 = 2$. So the solution of the IVP is $y = e^{-2t} + 2e^t - te^{-2t}$.

(ii) Using systems: Let $x_1 = y$ and $x_2 = y'$. Check that the given IVP yields the system $\vec{x}\,'(t) = A\vec{x}(t) + \vec{b}(t)$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 0 \\ 3e^{-2t} \end{bmatrix}$, with initial condition $\vec{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.
Solving the eigenvalue problem for A yields eigenpairs $\lambda_1 = -2$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\lambda_2 = 1$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So $X(t) = \begin{bmatrix} e^{-2t} & e^t \\ -2e^{-2t} & e^t \end{bmatrix}$ is a fundamental matrix for the homogeneous system $\vec{x}\,' = A\vec{x}$, and $\vec{x}_c = X(t)\vec{c}$ is the complementary solution of the original non-homogeneous system. Using variation-of-parameters in this context, we look for a solution of $\vec{x}\,' = A\vec{x} + \vec{b}$ of the form $\vec{x} = X(t)\vec{u}(t)$, which implies $X\vec{u}\,' = \vec{b}$ (see §7.6). This yields $\vec{u}\,' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -1 \\ e^{-3t} \end{bmatrix}$. So $\vec{u} = \begin{bmatrix} -t \\ -\frac{1}{3}e^{-3t} \end{bmatrix}$, and $\vec{x}_p = X(t)\vec{u}(t) = \begin{bmatrix} e^{-2t} & e^t \\ -2e^{-2t} & e^t \end{bmatrix}\begin{bmatrix} -t \\ -\frac{1}{3}e^{-3t} \end{bmatrix}$ is a particular solution. The general solution of $\vec{x}\,' = A\vec{x} + \vec{b}$ is $\vec{x} = \vec{x}_c + \vec{x}_p = X(t)\vec{c} + X(t)\vec{u}(t) = X(t)(\vec{c} + \vec{u}(t))$.
Using the initial condition, we get $X(0)(\vec{c} + \vec{u}(0)) = \vec{x}(0)$, i.e., $\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 - \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Solving yields $\vec{c} = \begin{bmatrix} \frac{4}{3} \\ 2 \end{bmatrix}$. So the solution of the IVP involving the system is $\vec{x} = X\vec{c} + X\vec{u} = \begin{bmatrix} e^{-2t} + 2e^t - te^{-2t} \\ -3e^{-2t} + 2e^t + 2te^{-2t} \end{bmatrix}$.
Since the first component of $\vec{x}$ is $x_1 = y$, we get $y = e^{-2t} + 2e^t - te^{-2t}$ as in part (i).

(iii) Using the Laplace transform:

$$L[y''] + L[y'] - 2L[y] = 3L[e^{-2t}] \qquad s^2L[y] - sy(0) - y'(0) + sL[y] - y(0) - 2L[y] = \frac{3}{s+2} \qquad \text{let } Y = L[y]$$

$$(s^2 + s - 2)Y - 3s - 2 = \frac{3}{s+2} \qquad Y = \frac{3s+2}{s^2+s-2} + \frac{3}{(s+2)(s^2+s-2)} = \frac{2}{s-1} + \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

So $y = L^{-1}\left[\frac{2}{s-1} + \frac{1}{s+2} - \frac{1}{(s+2)^2}\right] = 2e^t + e^{-2t} - te^{-2t}$ as in parts (i) and (ii).

3. Solve the IVP $y'' - 6y' + 25y = 0$, $y(0) = 8$, $y'(0) = 0$ (with and without using the Laplace transform).
Solve the IVP $\vec{x}\,' = A\vec{x}$, $\vec{x}(0) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$, where $A = \begin{bmatrix} 0 & 1 \\ -25 & 6 \end{bmatrix}$.

Answers: $y = 8e^{3t}\cos(4t) - 6e^{3t}\sin(4t)$ $\qquad \vec{x} = \begin{bmatrix} 8e^{3t}\cos(4t) - 6e^{3t}\sin(4t) \\ -50e^{3t}\sin(4t) \end{bmatrix}$