MATH 2090 — Systems with Defective Coefficient Matrix — Fall 2001

These examples illustrate the determination of a fundamental set of solutions for the system $\vec{x}\,' = A\vec{x}$ in instances where the matrix A is defective. We will consider only the case of an eigenvalue of algebraic multiplicity 3 here. A more general discussion may be found in §8.6 of the text. If λ is an eigenvalue of A of algebraic multiplicity 3, there are three possibilities:

(a) The geometric multiplicity of λ may also be 3. In this case, we can find three linearly independent eigenvectors corresponding to the eigenvalue λ. If $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are three such eigenvectors, we obtain three linearly independent solutions, $\vec{x}_1(t) = e^{\lambda t}\vec{v}_1$, $\vec{x}_2(t) = e^{\lambda t}\vec{v}_2$, and $\vec{x}_3(t) = e^{\lambda t}\vec{v}_3$ of the system $\vec{x}\,' = A\vec{x}$.

(b) The geometric multiplicity of λ may be 2. In this case, we can find two linearly independent eigenvectors, say $\vec{v}_1$ and $\vec{v}_2$, corresponding to the eigenvalue λ. From these, we obtain two linearly independent solutions, $\vec{x}_1(t) = e^{\lambda t}\vec{v}_1$ and $\vec{x}_2(t) = e^{\lambda t}\vec{v}_2$, of the system $\vec{x}\,' = A\vec{x}$. A third solution of this system is of the form $\vec{x}(t) = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}$, where $(A-\lambda I)\vec{v} = \vec{0}$ and $(A-\lambda I)\vec{w} = \vec{v}$. [Check this.] These conditions say that $\vec{v}$ is an eigenvector corresponding to λ, and $\vec{w}$ is an associated generalized eigenvector. The vector $\vec{v}$ may be expressed as a linear combination, $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ of the two eigenvectors we started with. Some care must be taken here: we must choose $\vec{v}$ so that the system of equations $(A - \lambda I)\vec{w} = \vec{v}$ is consistent.

§8.6 #6 $A = \begin{bmatrix} 15 & -32 & 12 \\ 8 & -17 & 6 \\ 0 & 0 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = -1$ and corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$

Solutions of $\vec{x}\,' = A\vec{x}$ are $\vec{x}_1(t) = e^{-t}\vec{v}_1 = e^{-t}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x}_2(t) = e^{-t}\vec{v}_2 = e^{-t}\begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$, and $\vec{x}_3(t) = te^{-t}\vec{v} + e^{-t}\vec{w}$. We must choose $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix} 2c_1 - 3c_2 \\ c_1 \\ 4c_2 \end{bmatrix}$ so that $(A+I)\vec{w} = \vec{v}$ is consistent. The augmented matrix of this system is $\left[\begin{array}{ccc|c} 16 & -32 & 12 & 2c_1 - 3c_2 \\ 8 & -16 & 6 & c_1 \\ 0 & 0 & 0 & 4c_2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 8 & -16 & 6 & c_1 \\ 0 & 0 & 0 & -3c_2 \\ 0 & 0 & 0 & 4c_2 \end{array}\right]$. So the system is consistent if $c_2 = 0$. If we take $c_1 = 1$, then $\vec{v} = \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and one solution of $(A+I)\vec{w} = \vec{v}$ is $\vec{w} = \begin{bmatrix} \frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$. Thus $\vec{x}_3(t) = te^{-t}\vec{v} + e^{-t}\vec{w} = te^{-t}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + e^{-t}\begin{bmatrix} \frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$ is a third solution of $\vec{x}\,' = A\vec{x}$. Checking that $W[\vec{x}_1, \vec{x}_2, \vec{x}_3] = e^{-3t}/2 \neq 0$, we have found a fundamental set of solutions.

(c) The geometric multiplicity of λ may be 1. In this case, we can find only one (independent) eigenvector, say $\vec{v}$, corresponding to the eigenvalue λ. From this, we obtain one solution, $\vec{x}_1(t) = e^{\lambda t}\vec{v}$, of the system $\vec{x}\,' = A\vec{x}$. Two additional solutions of this system are of the form $\vec{x}_2(t) = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}$ and $\vec{x}_3(t) = \frac{t^2}{2}e^{\lambda t}\vec{v} + te^{\lambda t}\vec{w} + e^{\lambda t}\vec{z}$, where $(A - \lambda I)\vec{v} = \vec{0}$, $(A - \lambda I)\vec{w} = \vec{v}$, and $(A - \lambda I)\vec{z} = \vec{w}$. [Check this.] These conditions say that $\vec{v}$ is an eigenvector corresponding to λ, and that $\vec{w}$ and $\vec{z}$ are associated generalized eigenvectors.

§8.6 #8 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 2 & -2 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and corresponding eigenvector $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Solutions of $\vec{x}\,' = A\vec{x}$ are $\vec{x}_1(t) = e^{t}\vec{v} = e^{-t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\vec{x}_2(t) = te^{t}\vec{v} + e^{t}\vec{w}$, and $\vec{x}_3(t) = \frac{t^2}{2}e^{t}\vec{v} + te^{t}\vec{w} + e^{t}\vec{z}$.

For the eigenvector $\vec{v}$ above, one solution of the system $(A-I)\vec{w} = \vec{v}$ is $\vec{w} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$. So $\vec{x}_2(t) = te^{t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + e^{t}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

For $\vec{w}$ as above, one solution of the system $(A-I)\vec{z} = \vec{w}$ is $\vec{z} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$. So $\vec{x}_3(t) = \dfrac{t^2}{2}e^{t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + te^{t}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + e^{t}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$.

Checking that $W[\vec{x}_1, \vec{x}_2, \vec{x}_3] = -e^{3t}/2 \neq 0$, we have found a fundamental set of solutions.