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A Weakly Over-Penalized Non-Symmetric Interior Penalty Method

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Abstract: A weakly over-penalized nonsymmetric interior penalty method for second order elliptic boundary value problems is considered in this paper. This method is consistent, stable for any choice of the penalty parameter and satisfies quasi-optimal error estimates in both the energy norm and the L_2 norm. Furthermore, there exists a simple block diagonal preconditioner that keeps the condition number of the discrete problem at the order of $O(h^{-2})$. Both theoretical and numerical results are presented.

Keywords: interior penalty methods, weakly over-penalized, preconditioner, quasi-optimal error estimates

Mathematics Subject Classification: 65N30

1 Introduction

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ and $f \in L_2(\Omega)$. In this paper we will consider a weakly over-penalized nonsymmetric interior penalty (WOPNIP) method for the following model problem: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^1(\Omega). \tag{1.1}$$

Before introducing our new method, we first review some well-known interior penalty methods for (1.1) to motivate our approach. Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω where h is the mesh size. We define V_h to be the discontinuous P_1 finite element space with respect to the triangulation $\mathcal{T}_h$. That is, $V_h = \{v \in L_2(\Omega) : \; v_T = v|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h\}$. Also, we define

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the jumps and means in the usual way [2, 3]. Let e be an interior edge shared by the triangles $T_1, T_2 \in \mathcal{T}_h$. Then we define on e,

$$[\![v]\!] = v_1\boldsymbol{n}_1 + v_2\boldsymbol{n}_2, \tag{1.2}$$

$$\{\!\{\nabla v\}\!\} = \frac{1}{2}\big(\nabla v_1 + \nabla v_2\big), \tag{1.3}$$

where $v_1 = v\big|_{T_1}$, $v_2 = v\big|_{T_2}$ and $\boldsymbol{n}_1$ (resp. $\boldsymbol{n}_2$) is the unit normal of e pointing towards the outside of T_1 (resp. T_2). On an edge e along $\partial\Omega$, we define

$$[\![v]\!] = (v\big|_e)\boldsymbol{n}, \tag{1.4}$$

$$\{\!\{\nabla v\}\!\} = (\nabla v)\big|_e, \tag{1.5}$$

where $\boldsymbol{n}$ is the unit normal of e pointing outside Ω.

The variational problem can be solved by the symmetric interior penalty Galerkin (SIPG) method [13, 1] and the nonsymmetric interior penalty (NIPG) method [12], which are defined as follows.

Find $u_h^{\pm} \in V_h$ such that

$$a_h^{\pm}(u_h^{\pm}, v) = \int_\Omega fv\,dx \qquad \forall\, v \in V_h, \tag{1.6}$$

where

$$a_h^{\pm}(w,v) = \sum_{T\in\mathcal{T}_h}\int_T \nabla w\cdot\nabla v\,dx - \sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla w\}\!\}\cdot[\![v]\!]\,ds \pm \sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla v\}\!\}\cdot[\![w]\!]\,ds + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|}\int_e [\![w]\!]\cdot[\![v]\!]\,ds,$$

$\mathcal{E}_h$ is the set of all the edges of $\mathcal{T}_h$, and $\eta > 0$ is a penalty parameter.

The function u_h^+ (resp. u_h^-) in (1.6) is the NIPG (resp. SIPG) approximate solution of (1.1). It is well-known that both methods are consistent, the NIPG method is stable for any choice of η, and the SIPG method is stable for sufficiently large η. Furthermore, when the methods are stable, we have [1, 12]

$$\|u - u_h^{\pm}\|_{a_h^+} \le Ch\|f\|_{L_2(\Omega)},$$

where

$$\|v\|_{a_h^+}^2 = a_h^+(v,v) = \sum_{T\in\mathcal{T}_h}\|\nabla v\|_{L_2(T)}^2 + \eta\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![v]\!]\|_{L_2(e)}^2,$$

provided

$$\eta \ge \eta_0 > 0. \tag{1.7}$$

From now on we assume that the penalty parameter η satisfies (1.7), i.e., it is bounded away from 0, and we will use C (with or without subscript) to denote a generic positive constant independent of f, h and η that can take different values at different occurrences.

Since the SIPG method is symmetric, it is also adjoint consistent. Consequently the Aubin-Nitsche duality argument can be applied and we have

$$\|u - u_h^-\|_{L_2(\Omega)} \le Ch^2\|f\|_{L_2(\Omega)}. \tag{1.8}$$

On the other hand the NIPG method is not adjoint consistent and the analog of (1.8) does not hold for u_h^+.

To recover the quasi-optimal L_2 error estimate for the NIPG approach, the following over-penalized method was introduced in [12].

Find $\tilde{u}_h \in V_h$ such that

$$\tilde{a}_h(\tilde{u}_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h, \tag{1.9}$$

where

$$\begin{aligned} \tilde{a}_h(w,v) = \sum_{T\in\mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx - \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla w\}\!\} \cdot [\![v]\!] \, ds + \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla v\}\!\} \cdot [\![w]\!] \, ds \\ + \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|^3} \int_e [\![w]\!] \cdot [\![v]\!] \, ds. \end{aligned}$$

The method (1.9) is stable for any $\eta > 0$ and

$$\|u - \tilde{u}_h\|_{L_2(\Omega)} + h\|u - \tilde{u}_h\|_{\tilde{a}_h} \le Ch^2 \|f\|_{L_2(\Omega)},$$

where $\|v\|_{\tilde{a}_h} = \sqrt{\tilde{a}_h(v,v)}$. Of course, the gain in the L_2 error estimate is at the expense of increasing the condition number of the discrete system from $O(h^{-2})$ for (1.6) to $O(h^{-4})$ for (1.9).

Our goal is to design an interior penalty method such that (i) it is consistent, (ii) it is stable for any choice of the penalty parameter, (iii) it satisfies quasi-optimal error estimates in both the energy norm and the L_2 norm, and (iv) we only have to solve a system of linear equations whose condition number is of order $O(h^{-2})$. Our idea for the new scheme is based on the following observation on the over-penalized method (1.9).

Since $\{\!\{\nabla w\}\!\}$ and $\{\!\{\nabla v\}\!\}$ are constant vectors along the edges of $\mathcal{T}_h$, we can rewrite the bilinear form as

$$\begin{aligned} \tilde{a}_h(w,v) = \sum_{T\in\mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx - \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla w\}\!\} \cdot \Pi_e^0 [\![v]\!] \, ds + \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla v\}\!\} \cdot \Pi_e^0 [\![w]\!] \, ds \\ + \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|^3} \int_e [\![w]\!] \cdot [\![v]\!] \, ds, \end{aligned}$$

where Π_e^0 is the orthogonal projection operator from $L_2(e)$ onto $P_0(e)$. That is,

$$\Pi_e^0 v = \frac{1}{|e|} \int_e v \, ds \qquad \forall\, v \in L_2(e). \tag{1.10}$$

Accordingly, we only need to over-penalize the integral $\int_e \Pi_e^0 [\![w]\!] \cdot \Pi_e^0 [\![v]\!] \, ds$. The resulting weakly over-penalized method is:

Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h, \tag{1.11}$$

where

$$\begin{aligned} a_h(w,v) = \sum_{T\in\mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx - \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla w\}\!\} \cdot [\![v]\!] \, ds \\ + \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla v\}\!\} \cdot [\![w]\!] \, ds + \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|^3} \int_e \Pi_e^0 [\![w]\!] \cdot \Pi_e^0 [\![v]\!] \, ds. \end{aligned} \tag{1.12}$$

Remark 1.1. Heuristically, the WOPNIP method (1.11)–(1.12) works because the weak over-penalization forces $\Pi_e^0 u_h$ to be almost 0 and it is well-known [8] that weakly continuous P_1 functions can be used to solve the Poisson problem (1.1). Also, at first glance the condition number of (1.11) is as bad as the condition number of (1.9). However, there is a simple block diagonal preconditioner (cf. Section 3 below) that will reduce the condition number of the system back to $O(h^{-2})$.

Remark 1.2. It follows from the midpoint rule that

$$\Pi_e^0 v = v(m_e) \qquad \forall\, v \in P_1(e), \tag{1.13}$$

where m_e is the midpoint of the edge e. Therefore the natural nodal basis for the WOPNIP method is associated with the midpoints of the edges of $\mathcal{T}_h$.

The rest of the paper is organized as follows. We derive quasi-optimal error estimates for the WOPNIP method in Section 2 and construct the block diagonal preconditioner in Section 3. We then present numerical results in Section 4 and end with some concluding remarks in Section 5.

2 Error Analysis

First we note that the solution u of (1.1) satisfies, via integration by parts,

$$a_h(u,v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h, \tag{2.1}$$

i.e., the scheme (1.11) is consistent. Furthermore, we have the elliptic regularity estimate [10]

$$\|u\|_{H^2(\Omega)} \le C\|f\|_{L_2(\Omega)}. \tag{2.2}$$

We will carry out the error analysis using the mesh-dependent norms $\|\cdot\|_h$ and $\|\!|\cdot|\!\|_h$ defined by

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} \|\nabla v\|_{L_2(T)}^2 + \sum_{e\in\mathcal{E}_h} |e|\, \|\{\!\{\nabla v\}\!\}\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|^3}\|\Pi_e^0[\![v]\!]\|_{L_2(e)}^2, \tag{2.3}$$

$$\|\!|v|\!\|_h^2 = \sum_{T\in\mathcal{T}_h} \|\nabla v\|_{L_2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|^3}\|\Pi_e^0[\![v]\!]\|_{L_2(e)}^2, \tag{2.4}$$

for all $v \in H^2(\Omega) + V_h$.

It is easy to see that

$$\|\!|v|\!\|_h^2 = a_h(v,v) \qquad \forall\, v \in H^2(\Omega) + V_h, \tag{2.5}$$

i.e., the scheme (1.11) is stable for any choice of the penalty parameter.

We begin our analysis with several lemmas, where we use the notation $\mathcal{T}_{h,e}$ to denote the set of the triangles of $\mathcal{T}_h$ that have $e \in \mathcal{E}_h$ as an edge.

Lemma 2.1. *The bilinear form $a_h(\cdot,\cdot)$ is bounded on $H^2(\Omega) + V_h$ with respect to $\|\cdot\|_h$, i.e,*

$$|a_h(w,v)| \le C\|w\|_h\|v\|_h \qquad \forall\, v, w \in H^2(\Omega) + V_h, \tag{2.6}$$

where the positive constant C depends only on the minimum angle of $\mathcal{T}_h$.

Proof. Let $e \in \mathcal{E}_h$ and $v \in H^2(\Omega) + V_h$ be arbitrary. We have

$$\frac{1}{|e|}\|[\![v]\!]\|_{L_2(e)}^2 \le C\Big(\frac{1}{|e|}\|\Pi_e^0[\![v]\!]\|_{L_2(e)}^2 + \frac{1}{|e|}\|[\![v]\!] - \Pi_e^0[\![v]\!]\|_{L_2(e)}^2\Big)$$

$$\leq C\frac{1}{|e|}\|\Pi_e^0[\![v]\!]\|_{L_2(e)}^2 + C\sum_{T\in\mathcal{T}_e}\frac{1}{|e|}\|v_T - \Pi_e^0 v_T\|_{L_2(e)}^2 \qquad (2.7)$$
$$\leq C\Big(\frac{1}{|e|}\|\Pi_e^0[\![v]\!]\|_{L_2(e)}^2 + \sum_{T\in\mathcal{T}_{h,e}}\|\nabla v\|_{L_2(T)}^2\Big),$$

where the constant C depends only on the shape of the triangles in $\mathcal{T}_{h,e}$. In the last step we have used the trace theorem (with scaling) and the Bramble-Hilbert lemma [4, 6].

The estimate (2.6) follows from the Cauchy-Schwarz inequality, (1.7), (1.12) and (2.7):

$$\begin{aligned}|a_h(w,v)| &\leq \sum_{T\in\mathcal{T}_h}\|\nabla w\|_{L_2(T)}\|\nabla v\|_{L_2(T)} + \sum_{e\in\mathcal{E}_h}\big(|e|^{1/2}\|\{\!\{\nabla w\}\!\}\|_{L_2(e)}\big)\big(|e|^{-1/2}\|[\![v]\!]\|_{L_2(e)}\big)\\ &\quad + \sum_{e\in\mathcal{E}_h}\big(|e|^{1/2}\|\{\!\{\nabla v\}\!\}\|_{L_2(e)}\big)\big(|e|^{-1/2}\|[\![w]\!]\|_{L_2(e)}\big) + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|^3}\|\Pi_e^0[\![w]\!]\|_{L_2(e)}\|\Pi_e^0[\![v]\!]\|_{L_2(e)}\\ &\leq \Big(\sum_{T\in\mathcal{T}_h}\|\nabla w\|_{L_2(T)}^2 + \sum_{e\in\mathcal{E}_h}|e|\|\{\!\{\nabla w\}\!\}\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\|[\![w]\!]\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|^3}\|\Pi_e^0[\![w]\!]\|_{L_2(e)}^2\Big)^{1/2}\\ &\quad\times\Big(\sum_{T\in\mathcal{T}_h}\|\nabla v\|_{L_2(T)}^2 + \sum_{e\in\mathcal{E}_h}|e|\|\{\!\{\nabla v\}\!\}\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\|[\![v]\!]\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|^3}\|\Pi_e^0[\![v]\!]\|_{L_2(e)}^2\Big)^{1/2}\\ &\leq C\|w\|_h\|v\|_h.\end{aligned}$$

□

It is clear from (2.3) and (2.4) that

$$|\!|\!|v|\!|\!|_h \leq \|v\|_h \qquad \forall\, v\in H^2(\Omega)+V_h. \qquad (2.8)$$

Moreover, the two norms are equivalent on V_h.

Lemma 2.2. *It holds that*

$$\|v\|_h \approx |\!|\!|v|\!|\!|_h \qquad \forall\, v\in V_h. \qquad (2.9)$$

Proof. Let $v\in V_h$ be arbitrary. It follows from (1.2), (1.4) and an inverse estimate that

$$\sum_{e\in\mathcal{E}_h}|e|\,\|\{\!\{\nabla v\}\!\}\|_{L_2(e)}^2 \leq C\sum_{e\in\mathcal{E}_h}|e|\sum_{T\in\mathcal{T}_{h,e}}\|\nabla v_T\|_{L_2(e)}^2 \leq C\sum_{T\in\mathcal{T}_h}\|\nabla v\|_{L_2(T)}^2,$$

where the constant C depends only on the minimum angle of $\mathcal{T}_h$. Therefore we have

$$\|v\|_h \leq C|\!|\!|v|\!|\!|_h \qquad \forall\, v\in V_h.$$

□

Lemma 2.3. *It holds that*

$$|\!|\!|u-u_h|\!|\!|_h \leq C\inf_{v\in V_h}\|u-v\|_h. \qquad (2.10)$$

Proof. It follows from (1.11) and (2.1) that

$$a_h(u-u_h,v) = 0 \qquad \forall\, v\in V_h. \qquad (2.11)$$

Using (2.5), (2.6), (2.9) and (2.11), we find

$$|\!|\!|u_h-v|\!|\!|_h^2 = a_h(u_h-v,u_h-v) = a_h(u-v,u_h-v) \leq C\|u-v\|_h|\!|\!|u_h-v|\!|\!|_h.$$

Therefore, we have

$$|||u_h - v|||_h \leq C\|u - v\|_h \qquad \forall\, v \in V_h,$$

and hence, in view of (2.8),

$$|||u - u_h|||_h \leq |||u - v|||_h + |||v - u_h|||_h \leq C\|u - v\|_h \qquad \forall\, v \in V_h,$$

which implies (2.10). □

Let Π_h be the nodal interpolation operator for the conforming P_1 finite element. It follows from (2.2), a standard interpolation error estimate [7, 6] and $[\![u]\!] = [\![\Pi_h u]\!] = 0$ that

$$|||u - \Pi_h u|||_h = \|\nabla(u - \Pi_h u)\|_{L_2(\Omega)} \leq Ch|u|_{H^2(\Omega)} \leq Ch\|f\|_{L_2(\Omega)}. \tag{2.12}$$

Furthermore, by the trace theorem (with scaling) and standard interpolation estimates, we have

$$\begin{aligned}\sum_{e\in\mathcal{E}_h} |e|\|\{\!\{\nabla(u - \Pi_h u)\}\!\}\|^2_{L_2(e)} &\leq C \sum_{e\in\mathcal{E}_h} \sum_{T\in\mathcal{T}_e} \Big(|u - \Pi_h u|^2_{H^1(T)} + h_T^2 |u|^2_{H^2(T)}\Big) \\ &\leq C \sum_{e\in\mathcal{E}_h} \sum_{T\in\mathcal{T}_e} h_T^2 |u|^2_{H^2(T)} \leq Ch^2|u|_{H^2(\Omega)}\end{aligned} \tag{2.13}$$

which, together with (2.2), (2.3) and (2.12), implies

$$\|u - \Pi_h u\|_h \leq Ch\|f\|_{L_2(\Omega)}. \tag{2.14}$$

Combining (2.10) and (2.14), we have the following result for the energy error.

Theorem 2.4. *It holds that*

$$|||u - u_h|||_h \leq Ch\|f\|_{L_2(\Omega)}. \tag{2.15}$$

We can also measure the error in the $\|\cdot\|_h$ norm.

Corollary 2.5. *It holds that*

$$\|u - u_h\|_h \leq Ch\|f\|_{L_2(\Omega)}. \tag{2.16}$$

Proof. From (2.9), (2.12), (2.14) and (2.15), we have

$$\begin{aligned}\|u - u_h\|_h &\leq \|u - \Pi_h u\|_h + \|\Pi_h u - u_h\|_h \\ &\leq Ch\|f\|_{L_2(\Omega)} + C|||\Pi_h u - u_h|||_h \\ &\leq Ch\|f\|_{L_2(\Omega)} + C\big(|||\Pi_h u - u|||_h + |||u - u_h|||_h\big) \leq Ch\|f\|_{L_2(\Omega)}.\end{aligned}$$

□

Finally, we can obtain an L_2 error estimate by a duality argument.

Lemma 2.6. *It holds that*

$$\|u - u_h\|_{L_2(\Omega)} \leq Ch\big(|||u - u_h|||_h + \|u - u_h\|_h\big). \tag{2.17}$$

Proof. Let $\phi \in H_0^1(\Omega)$ satisfy

$$\int_\Omega \nabla v \cdot \nabla\phi\, dx = \int_\Omega (u - u_h)v\, dx \qquad \forall\, v \in H_0^1(\Omega). \tag{2.18}$$

Then $\phi \in H^2(\Omega)$,

$$-\Delta\phi = u - u_h \qquad \text{in } \Omega, \tag{2.19}$$

and by elliptic regularity

$$\|\phi\|_{H^2(\Omega)} \leq C\|u - u_h\|_{L_2(\Omega)}. \tag{2.20}$$

Applying (2.13) and (2.14) to ϕ, we have

$$\|\phi - \Pi_h\phi\|_h \leq Ch\|u - u_h\|_{L_2(\Omega)}, \tag{2.21}$$

$$\sum_{e\in\mathcal{E}_h} |e|\|\{\!\{\nabla(\phi - \Pi_h\phi)\}\!\}\|^2_{L_2(e)} \leq Ch^2\|u - u_h\|^2_{L_2(\Omega)}. \tag{2.22}$$

It follows from (2.18), (2.19) and integration by parts that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\int_T \nabla(u - u_h)\cdot\nabla\phi\,dx &= \int_\Omega (u - u_h)u\,dx - \sum_{T\in\mathcal{T}_h}\int_T \nabla u_h\cdot\nabla\phi\,dx\\ &= \int_\Omega (u - u_h)^2\,dx - \sum_{e\in\mathcal{E}_h}\int_e [\![u_h]\!]\cdot\{\!\{\nabla\phi\}\!\}\,ds,\end{aligned}$$

and hence, in view of (1.12), (2.1) and the fact that $[\![\phi]\!] = 0 = [\![u]\!]$,

$$\begin{aligned}\|u - u_h\|^2_{L_2(\Omega)} &= \sum_{T\in\mathcal{T}_h}\int_T \nabla(u - u_h)\cdot\nabla\phi\,dx + \sum_{e\in\mathcal{E}_h}\int_e [\![u_h]\!]\cdot\{\!\{\nabla\phi\}\!\}\,ds\\ &= \sum_{T\in\mathcal{T}_h}\int_T \nabla(u - u_h)\cdot\nabla\phi\,dx - \sum_{e\in\mathcal{E}_h}\int_e [\![u - u_h]\!]\cdot\{\!\{\nabla\phi\}\!\}\,ds\\ &= a_h(u - u_h, \phi) - 2\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla\phi\}\!\}\cdot[\![u - u_h]\!]\,ds\\ &= a_h(u - u_h, \phi - \Pi_h\phi) - 2\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla(\phi - \Pi_h\phi)\}\!\}\cdot[\![u - u_h]\!]\,ds\\ &\qquad + 2\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla(\Pi_h\phi)\}\!\}\cdot[\![u - u_h]\!]\,ds.\end{aligned} \tag{2.23}$$

We now bound each of the three terms on the right-hand side of (2.23) separately. The first term can be bounded using (2.6) and (2.21):

$$a_h(u - u_h, \phi - \Pi_h\phi) \leq C\|u - u_h\|_h\|\phi - \Pi_h\phi\|_h \leq Ch\|u - u_h\|_h\|u - u_h\|_{L_2(\Omega)}. \tag{2.24}$$

The second term can be bounded using the Cauchy-Schwarz inequality, (1.7), (1.12), (2.20) and (2.22):

$$\begin{aligned}&\Big|\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla(\phi - \Pi_h\phi)\}\!\}\cdot[\![u - u_h]\!]\,ds\Big|\\ &\qquad\leq \sum_{e\in\mathcal{E}_h}\Big(\frac{|e|}{\eta}\Big)^{1/2}\|\{\!\{\nabla(\phi - \Pi_h\phi)\}\!\}\|_{L_2(e)}\Big(\frac{\eta}{|e|}\Big)^{1/2}\|[\![u - u_h]\!]\|_{L_2(e)}\\ &\qquad\leq \Big(\sum_{e\in\mathcal{E}_h}\frac{|e|}{\eta}\|\{\!\{\nabla(\phi - \Pi_h\phi)\}\!\}\|^2_{L_2(e)}\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|}\|[\![u - u_h)]\!]\|^2_{L_2(e)}\Big)^{1/2}\\ &\qquad\leq Ch\|u - u_h\|_{L_2(\Omega)}|\!|\!|u - u_h|\!|\!|_h.\end{aligned} \tag{2.25}$$

Finally, using the Cauchy-Schwarz inequality, (1.7), (1.12), (2.21) and a standard inverse estimate, we obtain

$$
\begin{aligned}
&\Big|\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla(\Pi_h\phi)\}\!\}\cdot[\![u-u_h]\!]\,ds\Big| \\
&\qquad = \Big|\sum_{e\in\mathcal{E}_h}\int_e \nabla(\Pi_h\phi)\cdot\Pi_e^0[\![u-u_h]\!]\,ds\Big| \\
&\qquad \le \sum_{e\in\mathcal{E}_h}\eta^{-1/2}|e|^{3/2}\|\nabla(\Pi_h\phi)\|_{L_2(e)}\,\eta^{1/2}|e|^{-3/2}\|\Pi_e^0[\![u-u_h]\!]\|_{L_2(e)} \\
&\qquad \le \Big(\sum_{e\in\mathcal{E}_h}\eta^{-1}|e|^3\|\nabla(\Pi_h\phi)\|^2_{L_2(e)}\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}\eta|e|^{-3}\|\Pi_e^0[\![u-u_h]\!]\|^2_{L_2(e)}\Big)^{1/2} \qquad (2.26)\\
&\qquad \le C\Big(h^2\sum_{T\in\mathcal{T}_h}\|\nabla(\Pi_h\phi)\|^2_{L_2(T)}\Big)^{1/2}|\!|\!|u-u_h|\!|\!|_h \\
&\qquad \le C\Big(h^2\sum_{T\in\mathcal{T}_h}\Big[\|\nabla(\phi-\Pi_h\phi)\|^2_{L_2(T)}+\|\nabla\phi\|^2_{L_2(T)}\Big]\Big)^{1/2}|\!|\!|u-u_h|\!|\!|_h \\
&\qquad \le Ch\|u-u_h\|_{L_2(\Omega)}|\!|\!|u-u_h|\!|\!|_h.
\end{aligned}
$$

The desired result follows from (2.23)–(2.26). □

Combining Theorem 2.4, Corollary 2.5 and Lemma 2.6, we have the following result for the L_2 error.

Theorem 2.7. *It holds that*

$$\|u-u_h\|_{L_2(\Omega)} \le Ch^2\|f\|_{L_2(\Omega)}. \qquad (2.27)$$

3 The Preconditioner

Let $A_h : V_h \longrightarrow V_h'$ be defined by

$$\langle A_h w, v\rangle = a_h(w,v) \qquad \forall\, v,w\in V_h, \qquad (3.1)$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $V_h'\times V_h$. In terms of A_h, the discrete problem (1.11) can be written as

$$A_h u_h = \phi_h, \qquad (3.2)$$

where $\phi_h \in V_h'$ is defined by

$$\langle\phi_h, v\rangle = (f,v)_{L_2(\Omega)} \qquad \forall\, v\in V_h.$$

The preconditioner for A_h is the operator $B_h : V_h \longrightarrow V_h'$ defined by

$$\langle B_h w, v\rangle = \sum_{T\in\mathcal{T}_h}\int_T wv\,dx + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|}\int_e \Pi_e^0[\![w]\!]\cdot\Pi_e^0[\![v]\!]\,ds \qquad \forall\, v,w\in V_h. \qquad (3.3)$$

Remark 3.1. It follows from a standard quadrature rule for quadratic functions that

$$\int_T wv\,dx = \frac{|T|}{3}\sum_{m\in\mathcal{M}_T} w(m)v(m) \qquad \forall\, w,v\in P_1(T), \qquad (3.4)$$

where $\mathcal{M}_T$ is the set of the three midpoints of T. Furthermore from (1.2), (1.4) and (1.13) we see that, for an interior edge e,

$$\frac{1}{|e|}\int_e \Pi_e^0[\![w]\!]\cdot\Pi_e^0[\![v]\!]\,ds = w_1(m_e)v_1(m_e)+w_2(m_e)v_2(m_e)-w_1(m_e)v_2(m_e)-w_2(m_e)v_1(m_e), \tag{3.5}$$

where $w_j = w\big|_{T_j}$ and $v_j = v\big|_{T_j}$ for $j = 1, 2$, and T_1 and T_2 are the two triangles in $\mathcal{T}_h$ sharing e as a common edge, and for a boundary edge e, we have

$$\frac{1}{|e|}\int_e \Pi_e^0[\![w]\!]\cdot\Pi_e^0[\![v]\!]\,ds = w(m_e)v(m_e). \tag{3.6}$$

Let $\mathbf{B}_h$ be the matrix representing B_h, i.e.,

$$\mathbf{v}^T\mathbf{B}_h\mathbf{w} = \langle B_h w, v\rangle, \tag{3.7}$$

where $\mathbf{v}$ (resp. $\mathbf{w}$) is the coordinate vector for v (resp. w) in V_h associated with the midpoints of the edges of $\mathcal{T}_h$. In view of Remark 3.1, the matrix $\mathbf{B}_h$ is block diagonal with 2×2 blocks (corresponding to the midpoints of interior edges) and 1×1 blocks (corresponding to the midpoints of boundary edges). Therefore it is trivial to compute $\mathbf{B}_h^{-1}$.

Let the operators $S_h, N_h : V_h \longrightarrow V_h'$ represent the symmetric and antisymmetric part of the bilinear form $a_h(\cdot,\cdot)$:

$$\langle S_h w, v\rangle = \sum_{T\in\mathcal{T}_h}\int_T \nabla w\cdot\nabla v\,dx + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|^3}\int_e \Pi_e^0[\![w]\!]\cdot\Pi_e^0[\![v]\!]\,ds \qquad \forall\, v, w\in V_h, \tag{3.8}$$

$$\langle N_h w, v\rangle = \sum_{e\in\mathcal{E}_h}\int_e \big(\{\!\{\nabla w\}\!\}[\![v]\!] - \{\!\{\nabla v\}\!\}[\![w]\!]\big)\,ds \qquad \forall\, v, w\in V_h. \tag{3.9}$$

It is clear from (3.1), (3.8) and (3.9) that

$$B_h^{-1}A_h = B_h^{-1}S_h - B_h^{-1}N_h. \tag{3.10}$$

Therefore we can estimate the condition number of the preconditioned system $B_h^{-1}A_h$ by examining the operators $B_h^{-1}S_h$ and $B_h^{-1}N_h$.

Lemma 3.2. *All of the eigenvalues of $B_h^{-1}S_h$ are real and the minimum and maximum eigenvalues of $B_h^{-1}S_h$ satisfy*

$$c \le \lambda_{\min}(B_h^{-1}S_h) \le \lambda_{\max}(B_h^{-1}S_h) \le Ch^{-2}, \tag{3.11}$$

where c and C are positive constants that depend only on the minimum angle of $\mathcal{T}_h$. In particular, we have

$$\kappa(B_h^{-1}S_h) = \frac{\lambda_{\max}(B_h^{-1}S_h)}{\lambda_{\min}(B_h^{-1}S_h)} \le Ch^{-2}. \tag{3.12}$$

Proof. Since the operator $B_h^{-1}S_h$ is symmetric with respect to the inner product $\langle B_h\cdot,\cdot\rangle$ on V_h, all the eigenvalues of $B_h^{-1}S_h$ are real, and it follows from the Raleigh quotient formula [9] that

$$\lambda_{\max}(B_h^{-1}S_h) = \max_{v\in V_h\setminus\{0\}}\frac{\langle S_h v, v\rangle}{\langle B_h v, v\rangle}, \tag{3.13}$$

$$\lambda_{\min}(B_h^{-1}S_h) = \min_{v\in V_h\setminus\{0\}}\frac{\langle S_h v, v\rangle}{\langle B_h v, v\rangle}. \tag{3.14}$$

Let $v \in V_h$ be arbitrary. By (3.3), (3.8), a standard inverse estimate and the fact that $|e| \approx h$, we have

$$\begin{aligned}\langle S_h v, v\rangle &= \sum_{T\in\mathcal{T}_h} \|\nabla v\|^2_{L_2(T)} + \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|^3}\|\Pi^0_e[\![v]\!]\|^2_{L_2(e)} \\ &\leq Ch^{-2}\sum_{T\in\mathcal{T}_h}\|v\|^2_{L_2(T)} + Ch^{-2}\sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|}\|\Pi^0_e[\![v]\!]\|^2_{L_2(e)} \leq Ch^{-2}\langle B_h v, v\rangle,\end{aligned}$$

which together with (3.13) implies that $\lambda_{\max}(B_h^{-1}S_h) \leq Ch^{-2}$.

In the other direction we have the following estimate from [5]:

$$\sum_{T\in\mathcal{T}_h}\|v\|^2_{L_2(T)} \leq C\Big(\sum_{T\in\mathcal{T}_h}\|\nabla v\|^2_{L_2(T)} + \sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\|[\![\Pi^0_e v]\!]\|^2_{L_2(e)}\Big). \tag{3.15}$$

Combining (1.7), (3.3), (3.8) and (3.15), we find

$$\begin{aligned}\langle B_h v, v\rangle &= \sum_{T\in\mathcal{T}_h}\|v\|^2_{L_2(T)} + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|}\|\Pi^0_e[\![v]\!]\|^2_{L_2(e)} \\ &\leq C\Big(\sum_{T\in\mathcal{T}_h}\|\nabla v\|^2_{L_2(T)} + \sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\|[\![\Pi^0_e v]\!]\|^2_{L_2(e)}\Big) + C\sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|^3}\|\Pi^0_e[\![v]\!]\|^2_{L_2(e)} \\ &\leq C\langle S_h v, v\rangle,\end{aligned}$$

which together with (3.14) implies $\lambda_{\min}(B_h^{-1}S_h) \geq c$. □

Lemma 3.3. *The eigenvalues of $B_h^{-1}N_h$ are purely imaginary and*

$$|\langle B_h^{-1}N_h w, v\rangle| \leq Ch^{-1}\langle B_h w, w\rangle^{1/2}\langle B_h v, v\rangle^{1/2} \qquad \forall\, w, v \in V_h, \tag{3.16}$$

where the positive constant C depends only on the minimum angle of $\mathcal{T}_h$. In particular, the spectral radius of $B_h^{-1}N_h$ satisfies

$$\rho(B_h^{-1}N_h) \leq Ch^{-1}. \tag{3.17}$$

Proof. Since the operator $B_h^{-1}N_h$ is antisymmetric with respect to the inner product $\langle B_h\cdot,\cdot\rangle$ on V_h, all the eigenvalues of $B_h^{-1}N_h$ are purely imaginary.

Using (1.7), (3.3), the Cauchy-Schwarz inequality and standard inverse estimates, we find

$$\begin{aligned}\Big|\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla w\}\!\}[\![v]\!]\,ds\Big| &= \Big|\sum_{e\in\mathcal{E}_h}\int_e \eta^{-1/2}|e|^{1/2}\{\!\{\nabla w\}\!\}\cdot \eta|e|^{-1/2}\Pi^0_e[\![v]\!]\,ds\Big| \\ &\leq \Big(\sum_{e\in\mathcal{E}_h}\eta^{-1}|e|\|\{\!\{\nabla w\}\!\}\|^2_{L_2(e)}\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}\eta|e|^{-1}\|\Pi^0_e[\![v]\!]\|^2_{L_2(e)}\Big)^{1/2} \\ &\leq C\Big(\sum_{T\in\mathcal{T}_h}\|\nabla w\|^2_{L_2(T)}\Big)^{1/2}\langle B_h v, v\rangle^{1/2} \\ &\leq C\Big(\sum_{T\in\mathcal{T}_h}h_T^{-2}\|w\|^2_{L_2(T)}\Big)^{1/2}\langle B_h v, v\rangle^{1/2} \\ &\leq Ch^{-1}\langle B_h w, w\rangle^{1/2}\langle B_h v, v\rangle^{1/2},\end{aligned}$$

and similarly

$$\Big|\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla v\}\!\}[\![w]\!]\,ds\Big| \leq Ch^{-1}\langle B_h v, v\rangle^{1/2}\langle B_h w, w\rangle^{1/2}.$$

□

We conclude from Lemma 3.2 (resp. Lemma 3.3) that $B_h^{-1}S_h$ (resp. $B_h^{-1}N_h$) behaves like a second (resp. first) order differential operator. In view of (3.10), the condition number of $B_h^{-1}A_h$ is also of order $O(h^{-2})$.

4 Numerical Results

Let Ω be the unit square $(0,1)\times(0,1)$ and the exact solution of (1.1) be given by

$$u(x,y) = xy(1-x)(1-y).$$

We solved (1.1) using the WOPNIP method (with $\eta = 0.1$, 1, 10 and 100) on uniform grids $\mathcal{T}_1,\ldots,\mathcal{T}_7$, where the length of a horizontal/vertical edge in $\mathcal{T}_k$ is $h_k = 2^{-k}$, and computed the relative errors

$$\frac{\sqrt{\sum_{T\in\mathcal{T}_k} \|\nabla(u-u_k)\|^2_{L_2(T)}}}{\|\nabla u\|_{L_2(\Omega)}}$$

in the piecewise H^1 semi-norm and the relative errors

$$\frac{\|u-u_k\|_{L_2(\Omega)}}{\|u\|_{L_2(\Omega)}}$$

in the L_2 norm. The results are plotted against k in Figure 1 and Figure 2. The error bounds (2.15) and (2.27) are clearly visible. Furthermore the relative errors for $\eta = 1$, 10 and 100 are eventually indistinguishable.

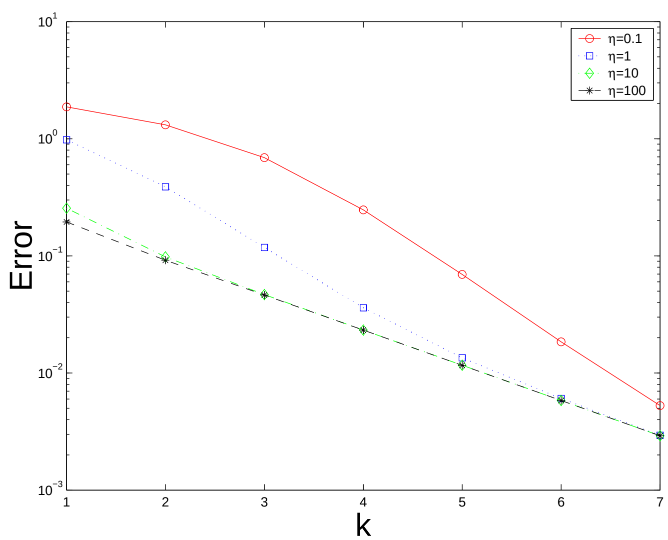

Figure 1: Relative errors in the piecewise H^1 semi-norm for $1 \le k \le 7$ and η=0.1, 1, 10, and 100

We also computed the condition number $\kappa(B_k^{-1}S_k)$ and the spectral radius $\rho(B_k^{-1}N_k)$ for $1 \le k \le 6$, and the numbers $h_k^2\kappa(B_k^{-1}S_k)$ and $h_k\rho(B_k^{-1}N_k)$ are tabulated in Table 1 and Table 2, which clearly demonstrate the estimates (3.12) and (3.17).

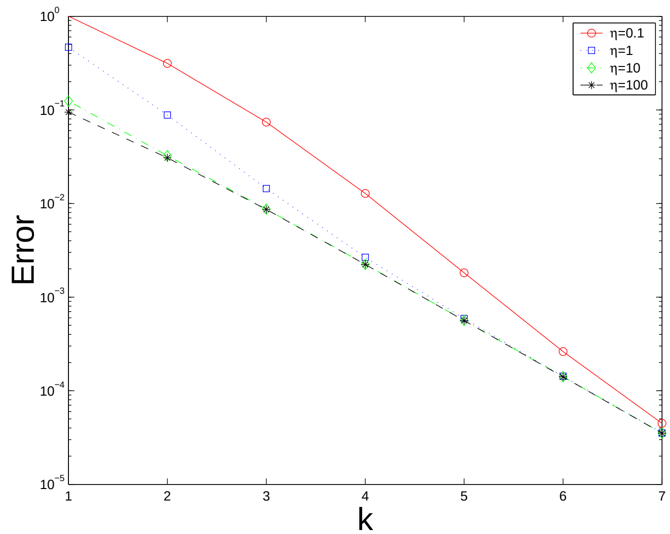

Figure 2: Relative errors in the L_2 norm for $1 \leq k \leq 7$ and η=0.1, 1, 10, and 100

k	$\eta = 0.1$	$\eta = 1$	$\eta = 10$	$\eta = 100$
1	24.3905	14.4633	15.3508	15.6675
2	7.0297	4.4732	4.3562	4.3492
3	2.9453	1.9869	1.8353	1.8180
4	2.0616	1.8471	1.8244	1.8221
5	1.8790	1.8289	1.8239	1.8233
6	1.8372	1.8250	1.8238	1.8237

Table 1: $h_k^2 \kappa(\mathcal{S}_k)$ for $1 \leq k \leq 6$ and η =0.1, 1, 10, 100

5 Concluding Remarks

The results in this paper can be extended to general second order elliptic boundary value problems. They can also be extended to nonconforming meshes with hanging nodes and higher order elements.

Multigrid algorithms for the WOPNIP method can be developed using a smoother built upon the block diagonal preconditioner in Section 3, and the quasi-optimal L_2 error estimate (2.27) is crucial for the convergence analysis of the multigrid algorithms.

These and other issues concerning the weakly over-penalized interior penalty methods will be addressed in [11].

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k	$\eta = 0.1$	$\eta = 1$	$\eta = 10$	$\eta = 100$
1	16.3927	5.8987	1.8941	0.5999
2	17.3113	5.6756	1.8016	0.5699
3	17.7000	5.6513	1.7888	0.5657
4	17.8400	5.6553	1.7888	0.5657
5	17.8764	5.6565	1.7888	0.5657
6	17.8855	5.6568	1.7889	0.5657

Table 2: $h_k \rho(\mathcal{N}_k)$ for $1 \leq k \leq 6$ and η =0.1, 1, 10, and 100

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