William A. Adkins, Mark G. Davidson

ORDINARY DIFFERENTIAL EQUATIONS
Student Solution Manual

January 23, 2012

Springer

Chapter 1
Solutions

Section 1.1

1. The rate of change in the population $P(t)$ is the derivative $P'(t)$. The Malthusian Growth Law states that the rate of change in the population is proportional to $P(t)$. Thus $P'(t) = kP(t)$, where k is the proportionality constant. Without reference to the t variable, the differential equation becomes $P' = kP$

3. Torricelli's law states that the change in height, $h'(t)$ is proportional to the square root of the height, $\sqrt{h(t)}$. Thus $h'(t) = \lambda\sqrt{h(t)}$, where λ is the proportionality constant.

5. The highest order derivative is y'' so the order is 2. The standard form is $y'' = t^3/y'$.

7. The highest order derivative is y'' so the order is 2. The standard form is $y'' = -(3y + ty')/t^2$.

9. The highest order derivative is $y^{(4)}$ so the order is 4. Solving for $y^{(4)}$ gives the standard form: $y^{(4)} = \sqrt[3]{(1 - (y''')^4)/t}$.

11. The highest order derivative is y''' so the order is 3. Solving for y''' gives the standard form: $y''' = 2y'' - 3y' + y$.

13. The following table summarizes the needed calculations:

Function	$ty'(t)$	$y(t)$
$y_1(t) = 0$	$ty_1'(t) = 0$	$y_1(t) = 0$
$y_2(t) = 3t$	$ty_2'(t) = 3t$	$y_2(t) = 3t$
$y_3(t) = -5t$	$ty_3'(t) = -5t$	$y_3(t) = -5t$
$y_4(t) = t^3$	$ty_4'(t) = 3t^3$	$y_4(t) = t^3$

To be a solution, the entries in the second and third columns need to be the same. Thus y_1, y_2, and y_3 are solutions.

15. The following table summarizes the needed calculations:

Function	$y'(t)$	$2y(t)(y(t)-1)$
$y_1(t) = 0$	$y_1'(t) = 0$	$2y_1(t)(y_1(t)-1) = 2\cdot 0\cdot(-1) = 0$
$y_2(t) = 1$	$y_2'(t) = 0$	$2y_2(t)(y_2(t)-1) = 2\cdot 1\cdot 0 = 0$
$y_3(t) = 2$	$y_3'(t) = 0$	$2y_3(t)(y_3(t)-1) = 2\cdot 2\cdot 1 = 4$
$y_4(t) = \frac{1}{1-e^{2t}}$	$y_4'(t) = \frac{2e^{2t}}{(1-e^{2t})^2}$	$2y_4(t)(y_4(t)-1) = 2\frac{1}{1-e^{2t}}\left(\frac{1}{1-e^{2t}}-1\right)$
		$= 2\frac{1}{1-e^{2t}}\frac{e^{2t}}{1-e^{2t}} = \frac{2e^{2t}}{(1-e^{2t})^2}$

Thus y_1, y_2, and y_4 are solutions.

17. The following table summarizes the needed calculations:

Function	$2y(t)y'(t)$	y^2+t-1
$y_1(t) = \sqrt{-t}$	$2\sqrt{-t}\frac{-1}{2\sqrt{-t}} = -1$	$(\sqrt{-t})^2 + t - 1 = -1$
$y_2(t) = -\sqrt{e^t - t}$	$-2\sqrt{e^t-t}\frac{-(e^t-1)}{2\sqrt{e^t-t}} = e^t - 1$	$(-\sqrt{e^t-t})^2 + t - 1 = e^t - 1$
$y_3(t) = \sqrt{t}$	$2\sqrt{t}\frac{1}{2\sqrt{t}} = 1$	$(\sqrt{t})^2 + t - 1 = 2t - 1$
$y_4(t) = -\sqrt{-t}$	$2(-\sqrt{-t})\frac{1}{2\sqrt{-t}} = -1$	$(-\sqrt{-t}))^2 + y - 1 = -1$

Thus y_1, y_2, and y_4 are solutions.

19.

$$\begin{aligned} y'(t) &= 3ce^{3t} \\ 3y+12 &= 3(ce^{3t}-4)+12 = 3ce^{3t}-12+12 = 3ce^{3t}. \end{aligned}$$

Note that $y(t)$ is defined for all $t \in \mathbb{R}$.

21.

$$\begin{aligned} y'(t) &= \frac{ce^t}{(1-ce^t)^2} \\ y^2(t) - y(t) &= \frac{1}{(1-ce^t)^2} - \frac{1}{1-ce^t} = \frac{1-(1-ce^t)}{(1-ce^t)^2} = \frac{ce^t}{(1-ce^t)^2}. \end{aligned}$$

If $c \le 0$ then the denominator $1 - ce^t > 0$ and $y(t)$ has domain $\mathbb{R}$. If $c > 0$ then $1 - ce^t = 0$ if $t = \ln \frac{1}{c} = -\ln c$. Thus $y(t)$ is defined either on the interval $(-\infty, -\ln c)$ or $(-\ln c, \infty)$.

23.

$$\begin{aligned} y'(t) &= \frac{-ce^t}{ce^t - 1} \\ -e^y - 1 &= -e^{-\ln(ce^t-1)} - 1 = \frac{-1}{ce^t-1} - 1 = \frac{-ce^t}{ce^t-1}. \end{aligned}$$

Since $c > 0$ then $y(t)$ is defined if and only if $ce^t - 1 > 0$. This occurs if $e^t > \frac{1}{t}$ which is true if $t > \ln \frac{1}{c} = -\ln c$. Thus $y(t)$ is defined on the interval $(-\ln c, \infty)$.

25.

$$\begin{aligned} y'(t) &= -(c-t)^{-2}(-1) = \frac{1}{(c-t)^2} \\ y^2(t) &= \frac{1}{(c-t)^2}. \end{aligned}$$

The denominator of $y(t)$ is 0 when $t = c$. Thus the two intervals where $y(t)$ is defined are $(-\infty, c)$ and (c, ∞).

27. Integration gives $y(t) = \frac{e^{2t}}{2} - t + c$.

29. Observe that $\frac{t+1}{t} = 1 + \frac{1}{t}$. Integration gives $y(t) = t + \ln|t| + c$.

31. We integrate two times. First, $y'(t) = -2\cos 3t + c_1$. Second, $y(t) = -\frac{2}{3}\sin 3t + c_1 t + c_2$.

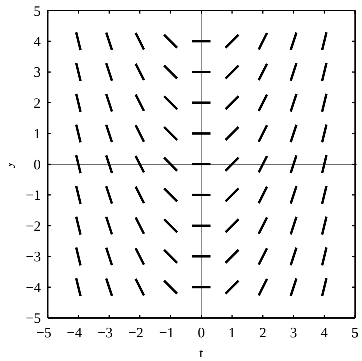
33. From Problem 20 the general solution is $y(t) = ce^{-t} + 3t - 3$. At $t = 0$ we get $0 = y(0) = ce^0 + 3(0) - 3 = c - 3$. It follows that $c = 3$ and $y(t) = 3e^{-t} + 3t - 3$.

35. From Problem 24 the general solution is $y(t) = c(t+1)^{-1}$. At $t = 1$ we get $-9 = y(1) = c(1+1)^{-1} = c/2$. It follows that $c = -18$ and $y(t) = -18(t+1)^{-1}$.

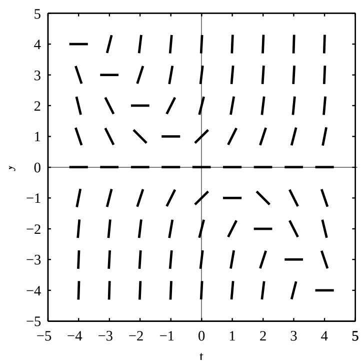
37. From Problem 28 the general solution is $y(t) = -te^{-t} - e^{-t} + c$. Evaluation at $t = 0$ gives $-1 = y(0) = -1 + c$ so $c = 0$. Hence $y(t) = -te^{-t} - e^{-t}$.

Section 1.2

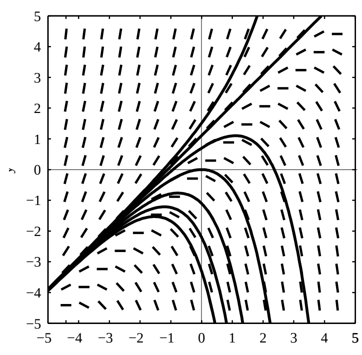
1. $$y' = t$$



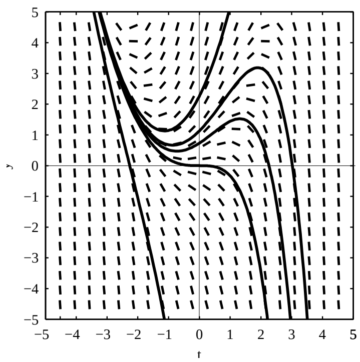
3. $$y' = y(y + t)$$



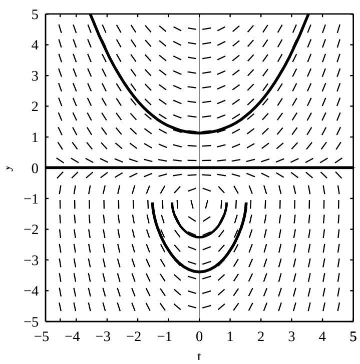
5.



7.



9.



11. We set $y(y+t)=0$. We look for constant solutions to $y(y+t)=0$, and we see that $y=0$ is the only constant (= equilibrium) solution.

13. The equation $1-y^2=0$ has two constant solutions: $y=1$ and $y=-1$

15. We substitute $y=at+b$ into $y'=\cos(t+y)$ to get $a=\cos((a+1)t+b)$. Equality for all t means that $\cos((a+1)t+b)$ must be a constant function, which can occur only if the coefficient of t is 0. This forces $a=-1$ leaving us with the equation $-1=\cos b$. This implies $b=(2n+1)\pi$, where n is an integer. Hence $y=-t+(2n+1)\pi$, $n\in\mathbb{Z}$ is a family of linear solutions.

17. Implicit differentiation with respect to t gives $2yy'-2t-3t^2=0$.

19. Differentiation gives $y'=3ct^2+2t$. However, from the given function we have $ct^3=y-t^2$ and hence $ct^2=\frac{y-t^2}{t}$. Substitution gives $y'=3\frac{y-t^2}{t}+2t=\frac{3y}{t}-t$.

Section 1.3

1. separable; $h(t) = 1$ and $g(y) = 2y(5 - y)$

3. First write in standard form: $y' = \frac{1-2ty}{t^2}$. We cannot write $\frac{1-2ty}{t^2}$ as a product of a function of t and a function of y. It is not separable.

5. Write in standard form to get: $y' = \frac{y-2yt}{t}$. Here we can write $\frac{y-2ty}{t} = y\frac{1-2t}{t}$. It is separable; $h(t) = \frac{1-2t}{t}$ and $g(y) = y$.

7. In standard form we get $y' = \frac{-2ty}{t^2+3y^2}$. We cannot write $y' = \frac{-2ty}{t^2+3y^2}$ as a product of a function of t and a function of y. It is not separable

9. In standard form we get: $y' = e^{-t}(y^3 - y)$ It is separable; $h(t) = e^{-t}$ and $g(y) = y^3 - y$

11. In standard form we get $y' = \frac{1-y^2}{ty}$. Clearly, $y = \pm 1$ are equilibrium solutions. Separating the variables gives

$$\frac{y}{1-y^2}\,dy = \frac{1}{t}\,dt.$$

Integrating both sides of this equation (using the substitution $u = 1-y^2$, $du = -2y\,dy$ for the integral on the left) gives

$$-\frac{1}{2}\ln|1-y^2| = \ln|t| + c.$$

Multiplying by -2, taking the exponential of both sides, and removing the absolute values gives $1 - y^2 = kt^{-2}$ where k is a nonzero constant. However, when $k = 0$ the equation becomes $1 - y^2 = 0$ and hence $y = \pm 1$. By considering an arbitrary constant (which we will call c), the implicit equation $t^2(1-y^2) = c$ includes the two equilibrium solutions for $c = 0$.

13. The variables are already separated, so integrate both sides to get $y^5/5 = t^2/2 + 2t + c$, c a real constant. Simplifying gives $y^5 = \frac{5}{2}t^2 + 10t + c$. We leave the answer in implicit form

15. In standard form we get $y' = (1-y)\tan t$ so $y = 1$ is a solution. Separating variables gives $\frac{dy}{1-y} = \tan t\,dt$. The function $\tan t$ is continuous on the interval $(-\pi/2, \pi/2)$ and so has an antiderivative. Integration gives $-\ln|1-y| = -\ln|\cos t| + k_1$. Multiplying by -1 and exponentiating gives $|1-y| = k_2\,|\cos t|$ where k_2 is a positive constant. Removing the absolute value signs gives $1 - y = k_3 \cos t$, with $k_3 \neq 0$. If we allow $k_3 = 0$ we get the equilibrium solution $y = 1$. Thus the solution can be written $y = 1 - c\cos t$, c any real constant.

17. There are two equilibrium solutions; $y = 0$ and $y = 4$. Separating variables and using partial fractions gives $\frac{1}{4}\left(\frac{1}{y} + \frac{1}{4-y}\right) dy = dt$. Integrating and simplifying gives $\ln\left|\frac{y}{4-y}\right| = 4t + k_1$ which is equivalent to $\frac{y}{4-y} = ce^{4t}$, c a nonzero constant. Solving for y gives $y = \frac{4ce^{4t}}{1+ce^{4t}}$. When $c = 0$ we get the equilibrium solution $y = 0$. However, there is no c which gives the other equilibrium solution $y = 4$.

19. Separating variables gives $\frac{dy}{y^2+1} = dt$ and integrating gives $\tan^{-1} y = t + c$. Thus $y = \tan(t + c)$, c a real constant.

21. In standard form we get $y' = \frac{-(y+1)}{y-1}\frac{1}{1+t^2}$ from which we see that $y = -1$ is an equilibrium solution. Separating variables and simplifying gives $\left(\frac{2}{y+1} - 1\right) dy = \frac{dt}{t^2+1}$. Integrating and simplifying gives $\ln(y + 1)^2 - y = \tan^{-1} t + c$.

23. The equilibrium solution is $y = 0$. Separating variables gives $y^{-2}\, dy = \frac{dt}{1-t}$. Integrating and simplifying gives $y = \frac{1}{\ln|1-t|+c}$, c real constant.

25. $y = 0$ is the only equilibrium solution. The equilibrium solution $y(t) = 0$ satisfies the initial condition $y(1) = 0$ so $y(t) = 0$ is the required solution.

27. In standard form we get $y' = -2ty$ so $y = 0$ is a solution. Separating variables and integrating gives $\ln|y| = -t^2 + k$. Solving for y gives $y = ce^{-t^2}$ and allowing $c = 0$ gives the equilibrium solution. The initial condition implies $4 = y(0) = ce^0 = c$. Thus $y = 4e^{-t^2}$.

29. Separating variables gives $\frac{dy}{y} = \frac{u}{u^2+1}\, du$ and integrating gives $\ln|y| = \ln\sqrt{u^2+1} + k$. Solving for y gives $y = c\sqrt{u^2+1}$, for $c \neq 0$. The initial condition gives $2 = y(0) = c$. So $y = 2\sqrt{u^2+1}$.

31. Since $y^2 + 1 \geq 1$ there are no equilibrium solutions. Separating the variables gives

$$\frac{dy}{y^2+1} = \frac{dt}{t^2},$$

and integration of both sides gives $\tan^{-1} y = -\frac{1}{t} + c$. Solve for y by applying the tangent function to both sides of the equation. Since $\tan(\tan^{-1} y) = y$, we get

$$y(t) = \tan\left(-\frac{1}{t} + c\right).$$

To find c observe that $\sqrt{3} = y(1) = \tan(-1 + c)$, which implies that $c - 1 = \pi/3$, so $c = 1 + \pi/3$. Hence

$$y(t) = \tan\left(-\frac{1}{t} + 1 + \frac{\pi}{3}\right).$$

To determine the maximum domain on which this solution is defined, note that the tangent function is defined on the interval $(-\pi/2, \pi/2)$, so that $y(t)$ is defined for all t satisfying

$$-\frac{\pi}{2} < -\frac{1}{t} + 1 + \frac{\pi}{3} < \frac{\pi}{2}.$$

The first inequality is solved to give $t > 6/(6+5\pi)$. The second equality is solved to give $t < 6/(6-\pi)$. Thus the maximum domain for the solution $y(t)$ is the interval $(a, b) = (6/(6+5\pi), 6/(6-\pi))$. $\lim_{t\to b^-} y(t) = \lim_{t\to b^-} \tan\left(-\frac{1}{t} + 1 + \frac{\pi}{3}\right) = \lim_{x\to\pi/2^-} \tan x = \infty$.

33. Let m denote the number of Argon-40 atoms in the sample. Then $8m$ is the number of Potassium-40 atoms. Let t be the age of the rock. Then t years ago there were $m + 8m = 9m$ atoms of Potassium-40. Hence $N(0) = 9m$. On the other hand, $8m = N(t) = N(0)e^{-\lambda t} = 9me^{-\lambda t}$. This implies that $\frac{8}{9} = e^{-\lambda t}$ and hence $t = \frac{-\ln\frac{8}{9}}{\lambda} = \frac{-\tau}{\ln 2}\ln\frac{8}{9} \approx 212$ million years old.

35. The ambient temperature is 32° F, the temperature of the ice water. From Equation (13) we get $T(t) = 32 + ke^{rt}$. At $t = 0$ we get $70 = 32 + k$, so $k = 38$ and $T(t) = 32 + 38e^{rt}$. After 30 minutes we have $55 = T(30) = 32 + 38e^{30r}$ and solving for r gives $r = \frac{1}{30}\ln\frac{23}{38}$. To find the time t when $T(t) = 45$ we solve $45 = 32 + 38e^{rt}$, with r as above. We get $t = 30\frac{\ln 13 - \ln 38}{\ln 23 - \ln 38} \approx 64$ minutes.

37. The ambient temperature is $T_a = 65°$. Equation (13) gives $T(t) = 65 + ke^{rt}$ for the temperature at time t. Since the initial temperature of the thermometer is $T(0) = 90$ we get $90 = T(0) = 65 + k$. Thus $k = 25$. The constant r is determined from the temperature at a second time: $85 = T(2) = 65 + 25e^{2r}$ so $r = \frac{1}{2}\ln\frac{4}{5}$. Thus $T(t) = 65 + 25e^{rt}$, with $r = \frac{1}{2}\ln\frac{4}{5}$. To answer the first question we solve the equation $75 = T(t) = 65 + 25e^{rt}$ for t. We get $t = 2\frac{\ln 2 - \ln 5}{\ln 4 - \ln 5} \approx 8.2$ minutes. The temperature at $t = 20$ is $T(20) = 65 + 25\left(\frac{4}{5}\right)^{10} \approx 67.7°$.

39. The ambient temperature is $T_a = 70°$. Equation (13) gives $T(t) = 70 + ke^{rt}$ for the temperature of the coffee at time t. We are asked to determine the initial temperature of the coffee so $T(0)$ is unknown. However, we have the equations

$$\begin{aligned} 150 = T(5) &= 70 + ke^{5r} \\ 142 = T(6) &= 70 + ke^{6r} \end{aligned}$$

or

$$\begin{aligned} 80 &= ke^{5r} \\ 72 &= ke^{4r}. \end{aligned}$$

Dividing the second equation by the first gives $\frac{72}{80} = e^r$ so $r = \ln 0.9$. From the first equation we get $k = 80e^{-5r} \approx 135.5$. We now calculate $T(0) = 70 + k \approx 205.5°$

41. Let us start time $t = 0$ at 1980. Then $P(0) = 290$. The Malthusian growth model gives $P(t) = 290e^{rt}$. At $t = 10$ (1990) we have $370 = 290e^{10r}$ and hence $r = \frac{1}{10}\ln\frac{37}{29}$. At $t = 30$ (2010) we have $P(30) = 290e^{30r} = 290\left(\frac{37}{29}\right)^3 \approx 602$.

43. We have $3P(0) = P(5) = P(0)e^{3r}$. So $r = \frac{\ln 3}{5}$. Now we solve the equation $2P(0) = P(t) = P(0)e^{rt}$ for t. We get $t = \frac{\ln 2}{r} = \frac{5\ln 2}{\ln 3} \approx 3.15$ years.

45. In the logistics equation $m = 5000$ and $P_0 = 2000$. Thus $P(t) = \frac{10{,}000{,}000}{2{,}000+3{,}000e^{-rt}} = \frac{10{,}000}{2+3e^{-rt}}$. Since $P(2) = 3000$ we get $3000 = \frac{10{,}000}{2+3e^{-rt}}$. Solving this equation for r gives $r = \ln\frac{3}{2}$. Now $P(4) = \frac{10{,}000}{2+3e^{-4r}} = \frac{10{,}000}{2+3\left(\frac{2}{3}\right)^4} \approx 3857$

47. We have $P(0) = P_0 = 400$, $P(3) = P_1 = 700$, and $P(6) = P_2 = 1000$. Using the result of the previous problem we get $m = \frac{700(700(400+1000)-2\cdot 400\cdot 1000)}{(700)^2-400\cdot 1000} = 1{,}400$

Section 1.4

1. This equation is already in standard form with $p(t) = 3$. An antiderivative of $p(t)$ is $P(t) = \int 3\,dt = 3t$ so the integrating factor is $\mu(t) = e^{3t}$. If we multiply the differential equation $y' + 3y = e^t$ by $\mu(t)$, we get the equation

$$e^{3t}y' + 3e^{3t}y = e^{4t},$$

and the left hand side of this equation is a perfect derivative, namely, $(e^{3t}y)'$. Thus, $(e^{3t}y)' = e^{4t}$. Now take antiderivatives of both sides and multiply by e^{-3t}. This gives

$$y = \frac{1}{4}e^t + ce^{-3t}$$

for the general solution of the equation. To find the constant c to satisfy the initial condition $y(0) = -2$, substitute $t = 0$ into the general solution to get $-2 = y(0) = \frac{1}{4} + c$. Hence $c = -\frac{9}{4}$, and the solution of the initial value problem is

$$y = \frac{1}{4}e^t - \frac{9}{4}e^{-3t}.$$

3. This equation is already in standard form. In this case $p(t) = -2$, an antiderivative is $P(t) = -2t$, and the integrating factor is $\mu(t) = e^{-2t}$. Now multiply by the integrating factor to get

$$e^{-2t}y' - 2e^{-2t}y = 1,$$

the left hand side of which is a perfect derivative $((e^{-2t})y)'$. Thus $((e^{-2t})y)' = 1$ and taking antiderivatives of both sides gives

$$(e^{-2t})y = t + c,$$

where $c \in \mathbb{R}$ is a constant. Now multiply by e^{2t} to get $y = te^{2t} + ce^{2t}$ for the general solution. Letting $t = 0$ gives $4 = y(0) = c$ so

$$y = te^{2t} + 4e^{2t}.$$

5. The general solution from Problem 4 is $y = \frac{e^t}{t} + \frac{c}{t}$. Now let $t = 1$ to get $0 = e + c$. So $c = -e$ and $y = \frac{e^t}{t} - \frac{e}{t}$.

7. We first put the equation in standard form and get

$$y' + \frac{1}{t}y = \cos(t^2).$$

In this case $p(t) = \frac{1}{t}$, an antiderivative is $P(t) = \ln t$, and the integrating factor is $\mu(t) = t$. Now multiply by the integrating factor to get

$$ty' + y = t\cos(t^2),$$

the left hand side of which is a perfect derivative $(ty)'$. Thus $(ty)' = t\cos(t^2)$ and taking antiderivatives of both sides gives $ty = \frac{1}{2}\sin(t^2) + c$ where $c \in \mathbb{R}$ is a constant. Now divide by t to get $y = \frac{\sin(t^2)}{2t} + \frac{c}{t}$. for the general solution.

9. In this case $p(t) = -3$ and the integrating factor is $e^{\int -3\,dt} = e^{-3t}$. Now multiply to get $e^{-3t}y' - 3e^{-3t}y = 25e^{-3t}\cos 4t$, which simplifies to $(e^{-3t}y)' = 25e^{-3t}\cos 4t$. Now integrate both sides to get $e^{-3t}y = (4\sin 4t - 3\cos 4t)e^{-3t} + c$, where we computed $\int 25e^{-3t}\cos 4t$ by parts twice. Dividing by e^{-3t} gives $y = 4\sin 4t - 3\cos 4t + ce^{3t}$.

11. In standard form we get $z' - 2tz = -2t^3$. An integrating factor is $e^{\int -2t\,dt} = e^{-t^2}$. Thus $(e^{-t^2}z)' = -2t^3e^{-t^2}$. Integrating both sides gives

$e^{-t^2}z = (t^2+1)e^{-t^2} + c$, where the integral of the right hand side is done by parts. Now divide by the integrating factor e^{-t^2} to get $z = t^2+1+ce^{t^2}$.

13. The given equation is in standard form, $p(t) = \cos t$, an antiderivative is $P(t) = -\sin t$, and the integrating factor is $\mu(t) = e^{-\sin t}$. Now multiply by the integrating factor to get

$$e^{-\sin t}y' + (\cos t)e^{-\sin t}y = (\cos t)e^{-\sin t},$$

the left hand side of which is a perfect derivative $((e^{-\sin t})y)'$. Thus

$$((e^{-\sin t})y)' = (\cos t)e^{-\sin t}$$

and taking antiderivatives of both sides gives $(e^{-\sin t})y = e^{-\sin t} + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by $e^{\sin t}$ to get $y = 1 + ce^{\sin t}$ for the general solution. To satisfy the initial condition, $0 = y(0) = 1 + ce^{\sin 0} = 1 + c$, so $c = -1$. Thus, the solution of the initial value problem is $y = 1 - e^{\sin t}$

15. The given linear differential equation is in standard form, $p(t) = \frac{-2}{t}$, an antiderivative is $P(t) = -2\ln t = \ln t^{-2}$, and the integrating factor is $\mu(t) = t^{-2}$. Now multiply by the integrating factor to get

$$t^{-2}y' - \frac{2}{t^3}y = \frac{t+1}{t^3} = t^{-2} + t^{-3},$$

the left hand side of which is a perfect derivative $(t^{-2}y)'$. Thus

$$(t^{-2}y)' = t^{-2} + t^{-3}$$

and taking antiderivatives of both sides gives $(t^{-2})y = -t^{-1} - \frac{t^{-2}}{2} + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by t^2 to and we get $y = -t - \frac{1}{2} + ct^{-2}$ for the general solution. Letting $t = 1$ gives $-3 = y(1) = \frac{-3}{2} + c$ so $c = \frac{-3}{2}$ and

$$y(t) = -t - \frac{1}{2} - \frac{3}{2}t^{-2}.$$

17. The given equation is in standard form, $p(t) = a$, $p(t) = a$, an antiderivative is $P(t) = at$, and the integrating factor is $\mu(t) = e^{at}$. Now multiply by the integrating factor to get $e^{at}y' + ae^{at}y = e^{(a+b)t}$, the left hand side of which is a perfect derivative $(e^{at}y)'$. Thus $(e^{at}y)' = e^{(a+b)t}$ and taking antiderivatives of both sides gives

$$(e^{at})y = \frac{1}{a+b}e^{(a+b)t} + c$$

where $c \in \mathbb{R}$ is a constant. Now multiply by e^{-at} to get

$$y = \frac{1}{a+b}e^{bt} + ce^{-at}$$

for the general solution.

19. In standard form we get $y' - (\tan t)y = \sec t$. In this case $p(t) = -\tan t$, an antiderivative is $P(t) = \ln \cos t$, and the integrating factor is $\mu(t) = e^{P(t)} = \cos t$. Now multiply by the integrating factor to get $(\cos t)y' - (\sin t)y = 1$, the left hand side of which is a perfect derivative $((\cos t)y)'$. Thus $((\cos t)y)' = 1$ and taking antiderivatives of both sides gives $(\cos t)y = t + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by $1/\cos t = \sec t$ and we get $y = (t + c)\sec t$ for the general solution.

21. The given differential equation is in standard form, $p(t) = -n/t$, an antiderivative is $P(t) = -n \ln t = \ln(t^{-n})$, and the integrating factor is $\mu(t) = t^{-n}$. Now multiply by the integrating factor to get $t^{-n}y' - nt^{-n-1}y = e^t$, the left hand side of which is a perfect derivative $(t^{-n}y)'$. Thus $(t^{-n}y)' = e^t$ and taking antiderivatives of both sides gives $(t^{-n})y = e^t + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by t^n to and we get $y = t^n e^t + ct^n$ for the general solution.

23. Divide by t to put the equation in the standard form

$$y' + \frac{3}{t}y = t.$$

In this case $p(t) = 3/t$, an antiderivative is $P(t) = 3\ln t = \ln(t^3)$, and the integrating factor is $\mu(t) = t^3$. Now multiply the standard form equation by the integrating factor to get $t^3y' + 3t^2y = t^4$, the left hand side of which is a perfect derivative $(t^3y)'$. Thus $(t^3y)' = t^4$ and taking antiderivatives of both sides gives $t^3y = \frac{1}{5}t^5 + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by t^{-3} and we get $y = \frac{1}{5}t^2 + ct^{-3}$ for the general solution. Letting $t = -1$ gives $2 = y(-1) = \frac{1}{5} - c$ so $c = \frac{-9}{5}$ and

$$y = \frac{1}{5}t^2 - \frac{9}{5}t^{-3}.$$

25. Divide by t^2 to put the equation in the standard form

$$y' + \frac{2}{t}y = t^{-2}.$$

In this case $p(t) = 2/t$, an antiderivative is $P(t) = 2\ln t = \ln t^2$, and the integrating factor is $\mu(t) = t^2$. Now multiply by the integrating factor to get $t^2y' + 2ty = 1$, the left hand side of which is a perfect derivative $(t^2y)'$. Thus $(t^2y)' = 1$ and taking antiderivatives of both sides gives $t^2y = t + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by t^{-2} to get $y = \frac{1}{t} + ct^{-2}$ for the general solution. Letting $t = 2$ gives $a = y(2) = \frac{1}{2} + \frac{c}{4}$ so $c = 4a - 2$

and

$$y = \frac{1}{t} + (4a-2)t^{-2}.$$

27. Let $V(t)$ denote the volume of fluid in the tank at time t. Initially, there are 10 gal of brine. For each minute that passes there is a net decrease of $4-3=1$ gal of brine. Thus $V(t) = 10-t$ gal.

input rate: input rate $= 3\ \frac{\text{gal}}{\text{min}} \times 1\ \frac{\text{lbs}}{\text{gal}} = 3\ \frac{\text{lbs}}{\text{min}}$.

output rate: output rate $= 4\ \frac{\text{gal}}{\text{min}} \times \frac{y(t)}{V(t)}\ \frac{\text{lbs}}{\text{gal}} = \frac{4y(t)}{10-t}\ \frac{\text{lbs}}{\text{min}}$.

Since $y' =$ input rate$-$output rate, it follows that $y(t)$ satisfies the initial value problem

$$y' = 3 - \frac{4}{10-t}y(t), \quad y(0) = 2.$$

Put in standard form, this equation becomes

$$y' + \frac{4}{10-t}y = 3.$$

The coefficient function is $p(t) = \frac{4}{10-t}$, $P(t) = \int p(t)\,dt = -4\ln(10-t) = \ln(10-t)^{-4}$, and the integrating factor is $\mu(t) = (10-t)^{-4}$. Multiplying the standard form equation by the integrating factor gives

$$((10-t)^{-4}y)' = 3(10-t)^{-4}.$$

Integrating and simplifying gives $y = (10-t) + c(10-t)^4$. The initial condition $y(0) = 2$ implies $2 = y(0) = 10 + c10^4$ and hence $c = -8/10^4$ so

$$y = (10-t) - \frac{8}{10^4}(10-t)^4.$$

Of course, this formula is valid for $0 \le t \le 10$. After 10 minutes there is no fluid and hence no salt in the tank.

29. Let $V(t)$ denote the volume of fluid in the container at time t. Initially, there are 10 L. For each minute that passes there is a net gain of $4-2=2$ L of fluid. So $V(t) = 10+2t$. The container overflows when $V(t) = 10+2t = 30$ or $t = 10$ minutes.

input rate: input rate $= 4\ \frac{\text{L}}{\text{min}} \times 20\ \frac{\text{g}}{\text{L}} = 80\ \frac{\text{g}}{\text{min}}$.

output rate: output rate $= 2\ \frac{\text{L}}{\text{min}} \times \frac{y(t)}{10+2t}\ \frac{\text{g}}{\text{L}} = \frac{2y(t)}{10+2t}\ \frac{\text{g}}{\text{min}}$.

Since $y' =$ input rate$-$output rate, it follows that $y(t)$ satisfies the initial value problem

$$y' = 80 - \frac{2y}{10+2t}, \quad y(0) = 0.$$

Simplifying and putting in standard form gives the equation

$$y' + \frac{1}{5+t}y = 80.$$

The coefficient function is $p(t) = \frac{1}{5+t}$, $P(t) = \int p(t)\,dt = \ln(5+t)$, and the integrating factor is $\mu(t) = 5+t$. Multiplying the standard form equation by the integrating factor gives $((5+t)y)' = 80(5+t)$. Integrating and simplifying gives $y = 40(5+t)+c(5+t)^{-1}$, where c is a constant. The initial condition $y(0) = 0$ implies $c = -1000$ so $y = 40(5+t) - 1000(5+t)^{-1}$. At the time the container overflows $t = 10$ we have $y(10) = 600 - \frac{1000}{15} \approx$ 533.33 g of salt.

31. input rate: input rate $= rc$

output rate: output rate $= r\frac{P(t)}{V}$

Let P_0 denote the amount of pollutant at time $t = 0$. Since $P' =$ input rate $-$ output rate it follows that $P(t)$ is a solution of the initial value problem

$$P' = rc - \frac{rP(t)}{V}, \quad P(0) = P_0.$$

Rewriting this equation in standard form gives the differential equation $P' + \frac{r}{V}P = rc$. The coefficient function is $p(t) = r/V$ and the integrating factor is $\mu(t) = e^{rt/V}$. Thus $(e^{\frac{rt}{V}}P)' = rce^{\frac{rt}{V}}$. Integrating and simplifying gives $P(t) = cV + ke^{\frac{-rt}{V}}$, where k is the constant of integration. The initial condition $P(0) = P_0$ implies $c = P_0 - cV$ so $P(t) = cV + (P_0 - cV)e^{\frac{-rt}{V}}$.

(a) $\lim_{t\to\infty} P(t) = cV$.

(b) When the river is cleaned up at $t = 0$ we assume the input concentration is $c = 0$. The amount of pollutant is therefore given by $P(t) = P_0 e^{\frac{-rt}{V}}$. This will reduce by 1/2 when $P(t) = \frac{1}{2}P_0$. We solve the equation $\frac{1}{2}P_0 = P_0 e^{\frac{-rt}{V}}$ for t and get $t_{1/2} = V\frac{\ln 2}{r}$. Similarly, the pollutant will reduce by 1/10 when $t_{1/10} = V\frac{\ln 10}{r}$.

(c) Letting V and r be given as stated for each lake gives:

Lake Erie: $t_{1/2} = 1.82$ years, $t_{1/10} = 6.05$ years.

Lake Ontario: $t_{1/2} = 5.43$ years, $t_{1/10} = 18.06$ years

33. Let $y_1(t)$ and $y_2(t)$ denote the amount of salt in Tank 1 and Tank 2, respectively, at time t. The volume of fluid at time t in Tank 1 is $V_1(t) = 10 + 2t$ and Tank 2 is $V_2(t) = 5 + t$.

input rate for Tank 1: input rate $= 4\ \frac{\text{L}}{\text{min}} \times 10\ \frac{\text{g}}{\text{L}} = 40\ \frac{\text{g}}{\text{min}}$.

output rate for Tank 1: output rate $= 2\ \frac{\text{L}}{\text{min}} \times \frac{y_1(t)}{10+2t}\ \frac{\text{g}}{\text{L}} = \frac{2y(t)}{10+2t}\ \frac{\text{g}}{\text{min}}$. The initial value problem for Tank 1 is thus

$$y_1' = 40 - \frac{2}{10+2t}y_1, \quad y_1(0) = 0.$$

Simplifying this equation and putting it in standard form gives

$$y_1' + \frac{1}{5+t}y_1 = 40.$$

The integrating factor is $\mu(t) = 5+t$. Thus $((5+t)y_1)' = 40(5+t)$. Integrating and simplifying gives $y_1(t) = 20(5+t) + c/(5+t)$. The initial condition $y(0) = 0$ implies $c = -500$ so $y_1 = 20(5+t) - 500/(5+t)$.

input rate for Tank 2: input rate $= 2\ \frac{\text{L}}{\text{min}} \times \frac{y_1(t)}{10+2t}\ \frac{\text{g}}{\text{L}} = 20 - \frac{500}{(5+t)^2}\ \frac{\text{g}}{\text{min}}$.

output rate for Tank 2: output rate $= 1\ \frac{\text{L}}{\text{min}} \times \frac{y_2(t)}{5+t}\ \frac{\text{g}}{\text{L}} = \frac{y_2(t)}{5+t}\ \frac{\text{g}}{\text{min}}$. The initial value problem for Tank 2 is thus

$$y_2' = 20 - 500/(5+t)^2 - \frac{1}{(5+t)}y_2, \quad y_2(0) = 0.$$

When this equation is put in standard form we get

$$y_2' + \frac{1}{(5+t)}y_2 = 20 - \frac{500}{(5+t)^2}.$$

The integrating factor is $\mu(t) = 5+t$. Thus

$$((5+t)y_2)' = 20(5+t) - \frac{500}{5+t}.$$

Integrating and simplifying gives

$$y_2(t) = 10(5+t) - \frac{500\ln(5+t)}{5+t} + \frac{c}{5+t}.$$

The initial condition $y_2(0) = 0$ implies $c = 500\ln 5 - 250$ so

$$y_2(t) = 10(5+t) - \frac{500\ln(5+t)}{5+t} + \frac{500\ln 5 - 250}{5+t}.$$

Section 1.5

1. In standard form we get $y' = \dfrac{y^2 + yt + t^2}{t^2}$ which is homogeneous since the degrees of the numerator and denominator are each two. Let $y = tv$. Then $v + tv' = v^2 + v + 1$ and so $tv' = v^2 + 1$. Separating variables gives $\dfrac{dv}{v^2+1} = \dfrac{dt}{t}$. Integrating gives $\tan^{-1} v = \ln|t| + c$. So $v = \tan(\ln|t| + c)$. Substituting $v = y/t$ gives $y = t\tan(\ln|t| + c)$. The initial condition implies $1 = y(1) = 1 \cdot \tan c = \tan c$ and hence $c = \pi/4$. Therefore $y(t) = t\tan(\ln|t| + \pi/4)$.

3. Since the numerator and denominator are homogeneous of degree 2 the quotient is homogeneous. Let $y = tv$. Then $v + tv' = v^2 - 4v + 6$. So $tv' = v^2 - 5v + 6 = (v-2)(v-3)$. There are two equilibrium solutions $v = 2, 3$. Separating the variables and using partial fractions gives $\left(\dfrac{1}{v-3} - \dfrac{1}{v-2}\right) dv = \dfrac{dt}{t}$. Integrating and simplifying gives $\ln\left|\dfrac{v-3}{v-2}\right| = \ln|t| + c$. Solving for v gives $v = \dfrac{3-2kt}{1-kt}$, for $k \neq 0$, and so $y = \dfrac{3t - 2kt^2}{1-kt}$, for $k \neq 0$. When $k = 0$ we get $v = 3$ or $y = 3t$, which is the same as the equilibrium solution $v = 3$. The equilibrium solution $v = 2$ gives $y = 2t$. Thus we can write the solutions as $y = \dfrac{3t-2kt^2}{1-kt}$, $k \in \mathbb{R}$ and $y = 2t$. The initial condition $y(2) = 4$ is satisfied for the linear equation $y = 2t$ but has no solution for the family $y = \dfrac{3t-2kt^2}{1-kt}$. Thus $y = 2t$ is the only solution.

5. Since the numerator and denominator are homogeneous of degree 2 the quotient is homogeneous. Let $y = tv$. Then $v + tv' = \dfrac{3v^2-1}{2v}$. Subtract v from both sides to get $tv' = \dfrac{v^2-1}{2v}$. The equilibrium solutions are $v = \pm 1$. Separating variables gives $\dfrac{2v\,dv}{v^2-1} = \dfrac{dt}{t}$ and integrating gives $\ln\left|v^2 - 1\right| = \ln|t| + c$. Exponentiating gives $v^2 - 1 = kt$ and by simplifying we get $v = \pm\sqrt{1+kt}$. Now $v = y/t$ so $y = \pm t\sqrt{1+kt}$. The equilibrium solutions $v = \pm 1$ become $y = \pm t$. These occur when $k = 0$, so are already included in the general formula.

7. In standard form we get $y' = \dfrac{y + \sqrt{t^2 - y^2}}{t}$. Since $\sqrt{(\alpha t)^2 - (\alpha y)^2} = \sqrt{\alpha^2(t^2 - y^2)} = \alpha\sqrt{t^2 - y^2}$ for $\alpha > 0$ it is easy to see that $y' = \dfrac{y + \sqrt{t^2-y^2}}{t}$ is homogeneous. Let $y = tv$. Then $v + tv' = v + \sqrt{1 - v^2}$.

Simplifying gives $tv' = \sqrt{1-v^2}$. Clearly $v = \pm 1$ are equilibrium solution. Separating variables gives $\dfrac{dv}{\sqrt{1-v^2}} = \dfrac{dt}{t}$. Integrating gives $\sin^{-1} v = \ln|t| + c$ and so $v = \sin(\ln|t| + c)$. Now substitute $v = y/t$ to get $y = t\sin(\ln|t| + c)$. The equilibrium solutions imply $y = \pm t$ are also solutions.

9. Note that although $y = 0$ is part of the general solution it does not satisfy the initial value. Divide both sides by y^2 to get $y^{-2}y' - y^{-1} = t$. Let $z = y^{-1}$. Then $z' = -y^{-2}y'$. Substituting gives $-z' - z = t$ or $z' + z = -t$. An integrating factor is e^t. So $(e^t z) = -te^t$. Integrating both sides gives $e^t z = -te^t + e^t + c$, where we have used integration by parts to compute $\int -te^t\,dt$. Solving for z gives $z = -t + 1 + ce^{-t}$. Now substitute $z = y^{-1}$ and solve for y to get $y = \dfrac{1}{-t+1+ce^{-t}}$. The initial condition implies $1 = \dfrac{1}{1+c}$ and so $c = 0$. The solution is thus $y = \dfrac{1}{1-t}$.

11. Note that $y = 0$ is a solution. First divide both sides by y^3 to get $y^{-3}y' + ty^{-2} = t$. Let $z = y^{-2}$. Then $z' = -2y^{-3}y'$, so $\dfrac{z'}{-2} = y^{-3}y'$. Substituting gives $\dfrac{z'}{-2} + tz = t$, which in standard form is $z' - 2tz = -2t$. An integrating factor is $e^{\int -2t\,dt} = e^{-t^2}$, so that $(e^{-t^2}z)' = -2te^{-t^2}$. Integrating both sides gives $e^{-t^2}z = e^{-t^2} + c$, where the integral of the right hand side is done by the substitution $u = -t^2$. Solving for z gives $z = 1 + ce^{t^2}$. Since $z = y^{-2}$ we find $y = \pm\dfrac{1}{\sqrt{1+ce^{t^2}}}$.

13. Note that $y = 0$ is a solution. Divide by y^2 and $(1-t^2)$ to get $y^{-2}y' - \dfrac{t}{1-t^2}y^{-1} = \dfrac{5t}{1-t^2}$. Let $z = y^{-1}$. Then $z' = -y^{-2}y'$ and substituting gives $-z' - \dfrac{t}{1-t^2}z = \dfrac{5t}{(1-t^2)}$. In standard form we get $z' + \dfrac{t}{1-t^2}z = \dfrac{-5t}{1-t^2}$. Multiplying by the integrating factor

$$\mu(t) = e^{\int \frac{t}{1-t^2}\,dt} = e^{-\frac{1}{2}\ln(1-t^2)} = (1-t^2)^{-1/2}$$

gives $(z(1-t^2)^{-1/2})' = -5t(1-t^2)^{-3/2}$. Integrating gives $z(1-t^2)^{-1/2} = -5(1-t^2)^{-1/2} + c$ and hence $z = -5 + c\sqrt{1-t^2}$. Since $z = y^{-1}$ we have $y = \dfrac{1}{-5+c\sqrt{1-t^2}}$.

15. If we divide by y we get $y' + ty = ty^{-1}$ which is a Bernoulli equation with $n = -1$. Note that since $n < 0$, $y = 0$ is *not* a solution. Dividing by y^{-1} gets us back to $yy' + ty^2 = t$. Let $z = y^2$. Then $z' = 2yy'$ so $\dfrac{z'}{2} + tz = t$

and in standard form we get $z' + 2tz = 2t$. An integrating factor is e^{t^2} so $(e^{t^2}z)' = 2te^{t^2}$. Integration gives $e^{t^2}z = e^{t^2} + c$ so $z = 1 + ce^{-t^2}$. Since $z = y^2$ we get $y = \pm\sqrt{1 + ce^{-t^2}}$. The initial condition implies $-2 = y(0) = -\sqrt{1+c}$ so $c = 3$. Therefore $y = -\sqrt{1 + 3e^{-t^2}}$.

17. Note that $y = 0$ is a solution. First divide both sides by y^3 to get $y^{-3}y' + y^{-2} = t$. Let $z = y^{-2}$. Then $z' = -2y^{-3}y'$. So $\dfrac{z'}{-2} + z = t$. In standard form we get $z' - 2z = -2t$. An integrating factor is $e^{\int -2\,dt} = e^{-2t}$ and hence $(e^{-2t}z)' = -2te^{-2t}$. Integration by parts gives $e^{-2t}z = (t + \dfrac{1}{2})e^{-2t} + c$ and hence $z = t + \dfrac{1}{2} + ce^{2t}$. Since $z = y^{-2}$ we get $y = \pm\dfrac{1}{\sqrt{t + \frac{1}{2} + ce^{2t}}}$.

19. Let $z = 2t - 2y + 1$. Then $z' = 2 - 2y'$ and so $y' = \dfrac{2 - z'}{2}$. Substituting we get $\dfrac{2 - z'}{2} = z^{-1}$ and in standard form we get $z' = 2 - 2z^{-1}$, a separable differential equation. Clearly, $z = 1$ is an equilibrium solution. Assume for now that $z \neq 1$. Then separating variables and simplifying using $1/(1 - z^{-1}) = \frac{z}{z-1} = 1 + \frac{1}{z-1}$ gives $\left(1 + \frac{1}{z-1}\right)\,dz = 2\,dt$. Integrating we get $z + \ln|z - 1| = 2t + c$. Now substitute $z = 2t - 2y + 1$ and simplify to get $-2y + \ln|2t - 2y| = c$, $c \in \mathbb{R}$. (We absorb the constant 1 in c.) The equilibrium solution $z = 1$ becomes $y = t$.

21. Let $z = t + y$. Then $z' = 1 + y'$ and substituting we get $z' - 1 = z^{-2}$. In standard form we get $z' = \dfrac{1 + z^2}{z^2}$. Separating variables and simplifying we get $\left(1 - \dfrac{1}{1 + z^2}\right)\,dz = dt$. Integrating we get $z - \tan^{-1} z = t + c$. Now let $z = t + y$ and simplify to get $y - \tan^{-1}(t + y) = c$, $c \in \mathbb{R}$.

23. This is the same as Exercise 16 where the Bernoulli equation technique there used the substitution $z = y^2$. Here use the given substitution to get $z' = 2yy' + 1$. Substituting we get $z' - 1 = z$ and in standard form $z' = 1 + z$. Clearly, $z = -1$ is an equilibrium solution. Separating variables gives $\dfrac{dz}{1 + z} = dt$ and integrating gives $\ln|1 + z| = t + c$, $c \in \mathbb{R}$. Solving for z we get $z = ke^t - 1$, where $k \neq 0$. Since $z = y^2 + t - 1$ we get $y^2 + t - 1 = ke^t - 1$ and solving for y gives $y = \pm\sqrt{ke^t - t}$. The case $k = 0$ gives the equilibrium solutions $y = \pm\sqrt{-t}$.

25. If $z = \ln y$ then $z' = \dfrac{y'}{y}$. Divide the given differential equation by y. Then $\dfrac{y'}{y} + \ln y = t$ and substitution gives $z' + z = t$. An integrating factor is e^t so $(e^t z)' = te^t$. Integration (by parts) gives $e^t z = (t-1)e^t + c$ and so $z = t - 1 + ce^{-t}$. Finally, solving for y we get $y = e^{t-1+ce^{-t}}$, $c \in \mathbb{R}$.

Section 1.6

1. This can be written in the form $M(t, y) + N(t, y)y' = 0$ where $M(t, y) = y^2 + 2t$ and $N(t, y) = 2ty$. Since $\partial M/\partial y = 2y = \partial N/\partial t$, the equation is exact, and the general solution is given implicitly by $V(t, y) = c$ where the function $V(t, y)$ is determined by the solution method for exact equations. Thus $V(t, y) = \int (y^2 + 2t)\, dt + \phi(y) = y^2 t + t^2 + \phi(y)$. The function $\phi(y)$ satisfies

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(y^2 t + t^2) + \frac{d\phi}{dy} = 2ty + \frac{d\phi}{dy} = N(t, y) = 2ty,$$

so that $d\phi/dy = 0$. Thus, $V(t, y) = y^2 t + t^2$ and the solutions to the differential equation are given implicitly by $t^2 + ty^2 = c$.

3. In this equation $M = 2t^2 - y$ and $N = t + y^2$. Since $\partial M/\partial y = -1$, while $\partial N/\partial t = 1$, the equation is not exact.

5. In this equation $M = 3y - 5t$ and $N = 2y - t$. Since $\partial M/\partial y = 3$, while $\partial N/\partial t = -1$, the equation is not exact.

7. This can be written in the form $M(t, y) + N(t, y)y' = 0$ where $M(t, y) = 2ty + 2t^3$ and $N(t, y) = t^2 - y$. Since $\partial M/\partial y = 2t = \partial N/\partial t$, the equation is exact, and the general solution is given implicitly by $V(t, y) = c$ where the function $V(t, y)$ is determined by the solution method for exact equations. Thus $V(t, y) = \int (2ty + 2t^3)\, dt + \phi(y) = t^2 y + t^4/2 + \phi(y)$. The function $\phi(y)$ satisfies

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(t^2 y + t^4/2) + \frac{d\phi}{dy} = t^2 + \frac{d\phi}{dy} = N(t, y) = t^2 - y,$$

so that $d\phi/dy = -y$. Hence $\phi(y) = -y^2/2$ so that $V(t, y) = t^2 y + t^4/2 - y^2/2$ and the solutions to the differential equation are given implicitly by $t^2 y + t^4/2 - y^2/2 = c$. Multiplying by 2 and completing the square (and replacing the constant $2c$ by c) gives $(y - t^2)^2 - 2t^4 = c$.

9. This can be written in the form $M(t, y) + N(t, y)y' = 0$ where $M(t, y) = -y$ and $N(t, y) = y^3 - t$. Since $\partial M/\partial y = -1 = \partial N/\partial t$, the equation is exact, and the general solution is given implicitly by $V(t, y) = c$ where the function $V(t, y)$ is determined by the solution method for exact equations. Thus $V(t, y) = \int (-y)\, dt + \phi(y) = -yt + \phi(y)$. The function $\phi(y)$ satisfies

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(-yt) + \frac{d\phi}{dy} = -t + \frac{d\phi}{dy} = N(t,\, y) = y^3 - t,$$

so that $d\phi/dy = y^3$. Hence, $\phi(y) = y^4/4$ so that $V(t,\, y) = y^4/4 - yt$ and the solutions to the differential equation are given implicitly by $y^4/4 - yt = c$.

Section 1.7

1. We first change the variable t to u and write $y'(u) = uy(u)$. Now integrate both sides from 1 to t to get $\int_1^t y'(u)\, du = \int_1^t uy(u)\, du$. Now the left side is $\int_1^t y'(u)\, du = y(t) - y(1) = y(t) - 1$. Thus $y(t) = 1 + \int_1^t uy(u)\, du$.

3. Change the variable t to u and write $y'(u) = \dfrac{u - y(u)}{u + y(u)}$. Now integrate both sides from 0 to t to get $\int_0^t y'(u)\, du = \int_0^t \dfrac{u - y(u)}{u + y(u)} du$. The left side is $y(t) - 1$ so $y(t) = 1 + \int_0^t \dfrac{u - y(u)}{u + y(u)}\, du$.

5. The corresponding integral equation is $y(t) = 1 + \int_1^t uy(u)\, du$. We then have

$$\begin{aligned}
y_0(t) &= 1 \\
y_1(t) &= 1 + \int_1^t u \cdot 1\, du = 1 + \left(\frac{u^2}{2}\right)\Bigg|_1^t = 1 + \frac{t^2}{2} - \frac{1}{2} = \frac{1 + t^2}{2} \\
y_2(t) &= 1 + \int_1^t u \left(\frac{1 + u^2}{2}\right)\, du = 1 + \left(\frac{u^2}{4} + \frac{u^4}{8}\right)\Bigg|_1^t = \frac{5}{8} + \frac{t^2}{4} + \frac{t^4}{8} \\
y_3(t) &= 1 + \int_1^t \left(\frac{5u}{8} + \frac{u^3}{4} + \frac{u^5}{8}\right)\, du = 1 + \left(\frac{5u^2}{16} + \frac{u^4}{16} + \frac{u^6}{48}\right)\Bigg|_1^t \\
&= \frac{29}{48} + \frac{5t^2}{16} + \frac{t^4}{16} + \frac{t^6}{48}.
\end{aligned}$$

7. The corresponding integral equation is $y(t) = \int_0^t (u + y^2(u))\, du$. We then have

$$\begin{aligned}
y_0(t) &= 0\\
y_1(t) &= \int_0^t (u+0)\,du = \frac{t^2}{2}\\
y_2(t) &= \int_0^t \left(u + \left(\frac{u^2}{2}\right)^2\right)\,du = \int_0^t \left(u+\frac{u^4}{4}\right)\,du = \frac{t^2}{2}+\frac{t^5}{20}\\
y_3(t) &= \int_0^t \left(u + \left(\frac{u^2}{2}+\frac{u^5}{20}\right)^2\right)\,du = \int_0^t \left(u+\frac{u^4}{4}+\frac{u^7}{20}+\frac{u^{10}}{400}\right)\,du\\
&= \frac{t^2}{2}+\frac{t^5}{20}+\frac{t^8}{160}+\frac{t^{11}}{4400}.
\end{aligned}$$

9. The corresponding integral equation is

$$y(t) = \int_0^t (1+(u-y(u))^2)\,du.$$

We then have

$$\begin{aligned}
y_0(t) &= 0\\
y_1(t) &= \int_0^t \left(1+(u-0)^2\right)\,du = \left(u+\frac{u^3}{3}\right)\Bigg|_0^t = t+\frac{t^3}{3}\\
y_2(t) &= \int_0^t \left(1+\left(u-\left(u+\frac{u^3}{3}\right)\right)^2\right)\,du = \int_0^t \left(1+\frac{u^6}{9}\right)\,du\\
&= \left(u+\frac{u^7}{63}\right)\Bigg|_0^t = t+\frac{t^7}{7\cdot 3^2}\\
y_3(t) &= \int_0^t \left(1+\frac{u^{14}}{7^2\cdot 3^4}\right)\,du = t+\frac{t^{15}}{15\cdot 7^2\cdot 3^4}\\
y_4(t) &= \int_0^t \left(1+\frac{u^{30}}{15^2\cdot 7^4\cdot 3^8}\right)\,du = t+\frac{t^{31}}{31\cdot 15^2\cdot 7^4\cdot 3^8}\\
y_5(t) &= \int_0^t \left(1+\frac{u^{62}}{31^2\cdot 15^4\cdot 7^8\cdot 3^16}\right)\,du = t+\frac{t^{63}}{63\cdot 31^2\cdot 15^4\cdot 7^8\cdot 3^{16}}
\end{aligned}$$

11. The right hand side is $F(t,\,y)=\sqrt{y}$. If $\mathcal{R}$ is any rectangle about $(1,0)$ then there are y-coordinates that are negative. Hence F is not defined on $\mathcal{R}$ and Picards' theorem does not apply.

13. The right hand side is $F(t,\,y)=\dfrac{t-y}{t+y}$. Then $F_y(t,\,y)=\dfrac{-2t}{(t+y)^2}$. Choose a rectangle $\mathcal{R}$ about $(0,-1)$ that contains no points on the line $t+y=0$. Then both F and F_y are continuous on $\mathcal{R}$. Picard's theorem applies and we can conclude there is a unique solution on an interval about 0.

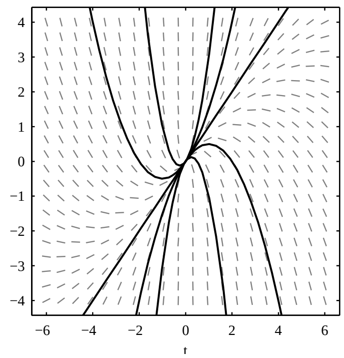
15. The corresponding integral equation is $y(t) = 1 + \int_0^t ay(u)\,du$. We thus have

$$\begin{aligned}
y_0(t) &= 1 \\
y_1(t) &= 1 + \int_0^t a\,du = 1 + at \\
y_2(t) &= 1 + \int_0^t a(1+au)\,du = 1 + \int_0^t (a + a^2u)\,du = 1 + at + \frac{a^2t^2}{2} \\
y_3(t) &= 1 + \int_0^t a\left(1 + au + \frac{a^2u^2}{2}\right)\,du = 1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{3!} \\
&\vdots \\
y_n(t) &= 1 + at + \frac{a^2t^2}{2} + \cdots + \frac{a^nt^n}{n!}.
\end{aligned}$$

We can write $y_n(t) = \sum_{k=0}^{n} \frac{a^kt^k}{k!}$. We recognize this sum as the first n terms of the Taylor series expansion for e^{at}. Thus the limiting function is $y(t) = \lim_{n\to\infty} y_n(t) = e^{at}$. It is straightforward to verify that it is a solution. If $F(t, y) = ay$ then $F_y(t, y) = a$. Both F and F_y are continuous on the whole (t, y)-plane. By Picard's theorem, Theorem 5, $y(t) = e^{at}$ is the only solution to the given initial value problem.

17. Let $F(t, y) = \cos(t + y)$. Then $F_y(t, y) = -\sin(t + y)$. Let y_1 and y_2 be arbitrary real numbers. Then by the mean value theorem there is a number y_0 in between y_1 and y_2 such that $|F(t, y_1) - F(t, y_2)| = |\sin(t + y_0)|\,|y_1 - y_2| \leq |y_1 - y_2|$. It follows that $F(t, y)$ is Lipschitz on any strip. Theorem 10 implies there is a unique solution on all of $\mathbb{R}$.

19. 1. First assume that $t \neq 0$. Then $ty' = 2y - t$ is linear and in standard form becomes $y' - 2y/t = -1$. An integrating factor is $\mu(t) = e^{\int(-2/t)\,dt} = t^{-2}$ and multiplying both sides by μ gives $t^{-2}y' - 2t^{-3}y = -t^{-2}$. This simplifies to $(t^{-2}y)' = -t^{-2}$. Now integrate to get $t^{-2}y = t^{-1} + c$ or $y(t) = t + ct^2$. We observe that this solution is also valid for $t = 0$. Graphs are given below for various values of c.



Graph of $y(t) = t + ct^3$ for various c

2. Every solution satisfies $y(0) = 0$. There is no contradiction to Theorem 5 since, in standard form, the equation is $y' = \dfrac{2}{t}y - 1 = F(t, y)$ and $F(t, y)$ is not continuous for $t = 0$.

21. No. Both $y_1(t)$ and $y_2(t)$ would be solutions to the initial value problem $y' = F(t, y)$, $y(0) = 0$. If $F(t, y)$ and $F_y(t, y)$ are both continuous near $(0, 0)$, then the initial value problem would have a unique solution by Theorem 5.

23. For $t < 0$ we have $y_1'(t) = 0$ and for $t > 0$ we have $y_1'(t) = 3t^2$. For $t = 0$ we calculate $y_1'(0) = \lim_{h\to 0} \dfrac{y_1(h) - y_1(0)}{h - 0} = \lim_{h\to 0} \dfrac{y_1(h)}{h}$. To compute this limit we show the left hand and right hand limits agree. We get

$$\begin{aligned} \lim_{h\to 0^+} \frac{y_1(h)}{h} &= \lim_{h\to 0^+} \frac{h^3}{h} = \lim_{h\to 0^+} h^2 = 0 \\ \lim_{h\to 0^-} \frac{y_1(h)}{h} &= \lim_{h\to 0^+} \frac{0}{h} = 0 \end{aligned}$$

It follows that $y_1'(t) = \begin{cases} 0, & \text{for } t < 0 \\ 3t^2 & \text{for } t \geq 0 \end{cases}$ and so

$$ty_1'(t) = \begin{cases} 0, & \text{for } t < 0 \\ 3t^3 & \text{for } t \geq 0 \end{cases}$$

On the other hand,

$$3y_1(t) = \begin{cases} 0, & \text{for } t < 0 \\ 3t^3 & \text{for } t \geq 0 \end{cases}$$

It follows that y_1 is a solution. It is trivial to see that $y_2(t)$ is a solution. There is no contraction to Theorem 5 since, in standard form $y' = \frac{3}{t}y = F(t, y)$ has a discontinuous $F(t, y)$ near $(0, 0)$. So Picard's theorem does not even apply.

Section 2.1

1. Apply the Laplace transform to both sides of the equation. For the left hand side we get $sY(s) - 2 - 4Y(s)$, while the right hand side is 0. Solve for $Y(s)$ to get $Y(s) = \frac{2}{s-4}$. From this we see that $y(t) = 2e^{4t}$.

3. Apply the Laplace transform to both sides of the equation. For the left hand side we get $sY(s) - 4Y(s)$, while the right hand side is $1/(s-4)$. Solve for $Y(s)$ to get $Y(s) = \frac{1}{(s-4)^2}$. Therefore, $y(t) = te^{4t}$.

5. Apply the Laplace transform to both sides of the equation. For the left hand side we get $sY(s) - 2 + 2Y(s)$, while the right hand side is $3/(s-1)$. Solve for $Y(s)$ to get

$$Y(s) = \frac{2}{s+2} + \frac{3}{(s-1)(s+2)} = \frac{1}{s+2} + \frac{1}{s-1}.$$

Thus $y(t) = e^{-2t} + e^{t}$.

7. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}\mathcal{L}\left\{y'' + 3y' + 2y\right\}(s) &= \mathcal{L}\left\{y''\right\}(s) + 3\mathcal{L}\left\{y'\right\}(s) + 2\mathcal{L}\left\{y\right\}(s) \\ &= s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) \\ &= (s^2 + 3s + 2)Y(s) - 3s - 3.\end{aligned}$$

Since the Laplace transform of 0 is 0, we now get

$$(s^2 + 3s + 2)Y(s) - 3s - 3 = 0.$$

Hence,

$$Y(s) = \frac{3s+3}{s^2+3s+2} = \frac{3(s+1)}{(s+1)(s+2)} = \frac{3}{s+2},$$

and therefore, $y(t) = 3e^{-2t}$.

9. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}\mathcal{L}\left\{y''+25y\right\}(s) &= \mathcal{L}\left\{y''\right\}(s)+25\mathcal{L}\left\{y\right\}(s)\\ &= s^2Y(s)-sy(0)-y'(0)+25Y(s)\\ &= (s^2+25)Y(s)-s+1.\end{aligned}$$

We now get

$$(s^2+25)Y(s)-s+1=0.$$

Hence,

$$Y(s)=\frac{s-1}{s^2+25}=\frac{s}{s^2+25}-\frac{1}{5}\frac{5}{s^2+25},$$

and therefore, $y(t)=\cos 5t-\frac{1}{5}\sin 5t$.

11. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}\mathcal{L}\left\{y''+8y'+16y\right\}(s) &= \mathcal{L}\left\{y''\right\}(s)+8\mathcal{L}\left\{y'\right\}(s)+16\mathcal{L}\left\{y\right\}(s)\\ &= s^2Y(s)-sy(0)-y'(0)+8(sY(s)-y(0))+16Y(s)\\ &= (s^2+8s+16)Y(s)-s-4.\end{aligned}$$

We now get

$$(s+4)^2Y(s)-(s+4)=0.$$

Hence,

$$Y(s)=\frac{s+4}{(s+4)^2}=\frac{1}{s+4}$$

and therefore $y(t)=e^{-4t}$

13. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}\mathcal{L}\left\{y''+4y'+4y\right\}(s) &= \mathcal{L}\left\{y''\right\}(s)+4\mathcal{L}\left\{y'\right\}(s)+4\mathcal{L}\left\{y\right\}(s)\\ &= s^2Y(s)-sy(0)-y'(0)+4(sY(s)-y(0))+4Y(s)\\ &= (s^2+4s+4)Y(s)-1.\end{aligned}$$

Since $\mathcal{L}\left\{e^{-2t}\right\}=1/(s+2)$ we get the algebraic equation

$$(s+2)^2Y(s)-1=\frac{1}{s+2}.$$

Hence,

$$Y(s)=\frac{1}{(s+2)^2}+\frac{1}{(s+2)^3}=\frac{1}{(s+2)^2}+\frac{1}{2}\frac{2}{(s+2)^3}$$

and therefore $y(t)=te^{-2t}+\frac{1}{2}t^2e^{-2t}$

Section 2.2

1.

$$\begin{aligned}
&\mathcal{L}\left\{3t+1\right\}(s)\\
=\;& \int_0^\infty (3t+1)e^{-st}\,dt\\
=\;& 3\int_0^\infty te^{-st}\,dt + \int_0^\infty e^{-st}\,dt\\
=\;& 3\left(\frac{t}{-s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt\right) + \frac{-1}{s}e^{-st}\Big|_0^\infty\\
=\;& 3\left(\left(\frac{1}{s}\right)\left(\frac{-1}{s}\right)e^{-st}\Big|_0^\infty\right) + \frac{1}{s}\\
=\;& \frac{3}{s^2} + \frac{1}{s}.
\end{aligned}$$

3.

$$\begin{aligned}
&\mathcal{L}\left\{e^{2t} - 3e^{-t}\right\}(s)\\
=\;& \int_0^\infty e^{-st}(e^{2t} - 3e^{-t})\,dt\\
=\;& \int_0^\infty e^{-st}e^{2t}\,dt - 3\int_0^\infty e^{-st}e^{-t}\,dt\\
=\;& \int_0^\infty e^{-(s-2)t}\,dt - 3\int_0^\infty e^{-(s+1)t}\,dt\\
=\;& \frac{1}{s-2} - \frac{3}{s+1}.
\end{aligned}$$

5. $\mathcal{L}\left\{5e^{2t}\right\} = 5\mathcal{L}\left\{e^{2t}\right\} = \dfrac{5}{s-2}$

7. $\mathcal{L}\left\{t^2 - 5t + 4\right\} = \mathcal{L}\left\{t^2\right\} - 5\mathcal{L}\left\{t\right\} + 4\mathcal{L}\left\{1\right\} = \dfrac{2}{s^3} - \dfrac{5}{s^2} + \dfrac{4}{s}$

9. $\mathcal{L}\left\{e^{-3t} + 7te^{-4t}\right\} = \mathcal{L}\left\{e^{-3t}\right\} + 7\mathcal{L}\left\{te^{-4t}\right\} = \dfrac{1}{s+3} + \dfrac{7}{(s+4)^2}$

11. $\mathcal{L}\left\{\cos 2t + \sin 2t\right\} = \mathcal{L}\left\{\cos 2t\right\} + \mathcal{L}\left\{\sin 2t\right\} = \dfrac{s}{s^2+2^2} + \dfrac{2}{s^2+2^2} = \dfrac{s+2}{s^2+4}$

13. $\mathcal{L}\left\{(te^{-2t})^2\right\}(s) = \mathcal{L}\left\{t^2e^{-4t}\right\}(s) = \dfrac{2}{(s+4)^3}$

15. $\mathcal{L}\left\{(t+e^{2t})^2\right\}(s) = \mathcal{L}\left\{t^2+2te^{2t}+e^{4t}\right\}(s) = \mathcal{L}\left\{t^2\right\}(s) + 2\mathcal{L}\left\{te^{2t}\right\}(s) + \mathcal{L}\left\{e^{4t}\right\}(s) = \dfrac{2}{s^3} + \dfrac{2}{(s-2)^2} + \dfrac{1}{s-4}$

17. $\mathcal{L}\left\{\dfrac{t^4}{e^{4t}}\right\}(s) = \mathcal{L}\left\{t^4e^{-4t}\right\}(s) = \dfrac{4!}{(s+4)^5} = \dfrac{24}{(s+4)^5}$

19. $\mathcal{L}\left\{te^{3t}\right\}(s) = -\left(\mathcal{L}\left\{e^{3t}\right\}\right)'(s) = -\left(\dfrac{1}{s-3}\right)' = \dfrac{1}{(s-3)^2}$

21. Here we use the transform derivative principle twice to get $\mathcal{L}\left\{t^2\sin 2t\right\}(s) = (\mathcal{L}\left\{\sin 2t\right\})'' = \left(\dfrac{2}{s^2+4}\right)'' = \left(\dfrac{-4s}{(s^2+4)^2}\right)' = \dfrac{12s^2-16}{(s^2+4)^3}$

23. $\mathcal{L}\left\{tf(t)\right\}(s) = -\mathcal{L}\left\{f(t)\right\}'(s) = -\left(\ln\left(\dfrac{s^2}{s^2+1}\right)\right)' = \dfrac{2s}{s^2+1} - \dfrac{2}{s}$

25. $\mathcal{L}\left\{\mathrm{Ei}(6t)\right\}(s) = \dfrac{1}{6}\,\mathcal{L}\left\{\mathrm{Ei}(t)\right\}(s)\big|_{s\mapsto s/6} = \dfrac{1}{6}\dfrac{\ln((s/6)+1)}{s/6} = \dfrac{\ln(s+6)-\ln 6}{s}$

27. We use the identity $\sin^2\theta = \dfrac{1}{2}(1-\cos 2\theta)$. $\mathcal{L}\left\{\sin^2 bt\right\}(s) = \dfrac{1}{2}\mathcal{L}\left\{1-\cos 2bt\right\}(s) = \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4b^2}\right) = \dfrac{2b^2}{s(s^2+4b^2)}$;

29. We use the identity $\sin at\cos bt = \dfrac{1}{2}\left(\sin(a+b)t + \sin(a-b)t\right)$.

$$\begin{aligned}\mathcal{L}\left\{\sin at\cos bt\right\} &= \frac{1}{2}\left(\mathcal{L}\left\{\sin(a+b)t\right\} + \mathcal{L}\left\{\sin(a-b)t\right\}\right)\\ &= \frac{1}{2}\left(\frac{a-b}{s^2+(a-b)^2} + \frac{a+b}{s^2+(a+b)^2}\right).\end{aligned}$$

31. $\mathcal{L}\left\{\sinh bt\right\} = \dfrac{1}{2}\left(\mathcal{L}\left\{e^{bt}-e^{-bt}\right\}\right) = \dfrac{1}{2}\left(\dfrac{1}{s+b} - \dfrac{1}{s-b}\right) = \dfrac{b}{s^2-b^2}$

33. Let $f(t) = \sinh bt$. Then $f'(t) = b\cosh t$ and $f''(t) = b^2\sinh t$. Further, $f(t)|_{t=0} = 0$ and $f'(t)|_{t=0} = b$. Thus $b^2\mathcal{L}\left\{\sinh bt\right\} = \mathcal{L}\left\{f''(t)\right\} = s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0) = s^2\mathcal{L}\left\{f(t)\right\} - b$. Solving for $\mathcal{L}\left\{f(t)\right\}$ gives $\mathcal{L}\left\{\sinh bt\right\} = \dfrac{b}{s^2-b^2}$.

35. Let $g(t) = \int_0^t f(u)\,du$ and note that $g'(t) = f(t)$ and $g(0) = \int_0^0 f(u)\,du = 0$. Now apply the input derivative formula to $g(t)$, to get

$$F(s) = \mathcal{L}\left\{f(t)\right\}(s) = \mathcal{L}\left\{g'(t)\right\}(s) = s\mathcal{L}\left\{g(t)\right\}(s) - g(0) = sG(s).$$

Solving for $G(s)$ gives $G(s) = F(s)/s$.

37. Suppose f is of exponential type of order a and g is of exponential type of order b. Then there are numbers K and L so that $|f(t)| \leq Ke^{at}$ and $|g(t)| \leq Le^{bt}$. Now $|f(t)g(t)| \leq Ke^{at}Le^{bt} = KLe^{(a+b)t}$. If follows that $f+g$ is of exponential type of order $a+b$.

39. Suppose a and K are real and $|y(t)| \leq Ke^{at}$. Then $y(t)e^{-at}$ is bounded by K. But

$$\begin{aligned} e^{t^2}e^{-at} &= e^{t^2-at+\frac{a^2}{4}}e^{-\frac{a^2}{4}} \\ &= e^{(t-\frac{a}{2})^2}e^{-\frac{a^2}{4}} \\ &= e^{u^2}e^{-\frac{a^2}{4}}, \end{aligned}$$

where $u = t - \frac{a}{2}$. As t approaches infinity so does u. Since $\lim_{u\to\infty} e^{u^2} = \infty$ it is clear that $\lim_{t\to\infty} e^{t^2}e^{-at} = \infty$, for all $a \in \mathbb{R}$, and hence $y(t)e^{-at}$ is not bounded. It follows that $y(t)$ is not of exponential type.

41. $y(t)$ is of exponential type because it is continuous and bounded. On the other hand, $y'(t) = \cos(e^{t^2})e^{t^2}(2t)$. Suppose there is a K and a so that $|y'(t)| \leq Ke^{at}$ for all $t \geq 0$. We need only show that there are some t for which this inequality does not hold. Since $\cos e^{t^2}$ oscillates between -1 and 1 let's focus on those t for which $\cos e^{t^2} = 1$. This happens when e^{t^2} is a multiple of 2π, i.e. $e^{t^2} = 2\pi n$ for some n. Thus $t = t_n = \sqrt{\ln(2\pi n)}$. If the inequality $|y'(t)| \leq Ke^{at}$ is valid for all $t \geq 0$ it is valid for t_n for all $n > 0$. We then get the inequality $2t_n e^{t_n^2} \leq Ke^{at_n}$. Now divide by e^{at_n}, combine, complete the square, and simplify to get the inequality $2t_n e^{(t_n-a/2)^2} \leq Ke^{a^2/4}$. Choose n so that $t_n > K$ and $t_n > a$. Then this last inequality is not satisfied. It follows that $y'(t)$ is not of exponential type. Now consider the definite integral $\int_0^M e^{-st}y'(t)\,dt$ and compute by parts: We get

$$\int_0^M e^{-st}y'(t)\,dt = y(t)e^{-st}\Big|_0^M + s\int_0^M e^{-st}y(t)\,dt.$$

Since $y(t) = \sin(e^{t^2})$ is bounded and $y(0) = 0$ it follows that

$$\lim_{M\to\infty} y(t)e^{-st}\Big|_0^M = 0.$$

Taking limits as $M \to \infty$ in the equation above gives $\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\}$. The righthand side exists because $y(t)$ is bounded.

(a) Show that $\Gamma(1) = 1$.

(b) Show that Γ satisfies the recursion formula $\Gamma(\beta+1) = \beta\Gamma(\beta)$. (*Hint*: Integrate by parts.)
(c) Show that $\Gamma(n+1) = n!$ when n is a nonnegative integer.

43. Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Then $dx\,dy = r\,dr\,d\theta$ and the domain of integration is the first quadrant of the plane, which in polar coordinates is given by $0 \le \theta \le \pi/2$, $0 \le r < \infty$. Thus

$$\begin{aligned} \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy &= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,dr\,d\theta \\ &= \frac{\pi}{2} \int_0^\infty e^{-r^2} r\,dr \\ &= \frac{\pi}{2} \left. -\frac{e^{-r^2}}{2} \right|_0^\infty = \frac{\pi}{4}. \end{aligned}$$

Hence, $I = \sqrt{\pi}/2$.

Section 2.3

1.

The $s-1$ -chain	
$\dfrac{5s+10}{(s-1)(s+4)}$	$\dfrac{3}{s-1}$
$\dfrac{2}{s+4}$	

3.

The $s-5$ -chain	
$\dfrac{1}{(s+2)(s-5)}$	$\dfrac{1/7}{(s-5)}$
$\dfrac{-1/7}{(s+2)}$	

5.

The $s-1$-chain	
$\frac{3s+1}{(s-1)(s^2+1)}$	$\frac{2}{s-1}$
$\frac{-2s+1}{s^2+1}$	

7.

The $s+3$-chain	
$\frac{s^2+s-3}{(s+3)^3}$	$\frac{3}{(s+3)^3}$
$\frac{s-2}{(s+3)^2}$	$\frac{-5}{(s+3)^2}$
$\frac{1}{s+3}$	$\frac{1}{s+3}$
0	

9.

The $s+1$-chain	
$\frac{s}{(s+2)^2(s+1)^2}$	$\frac{-1}{(s+1)^2}$
$\frac{s+4}{(s+2)^2(s+1)}$	$\frac{3}{s+1}$
$\frac{-3s-8}{(s+2)^2}$	

11.

The $s-5$ -chain	
$\frac{1}{(s-5)^5(s-6)}$	$\frac{-1}{(s-5)^5}$
$\frac{1}{(s-5)^4(s-6)}$	$\frac{-1}{(s-5)^4}$
$\frac{1}{(s-5)^3(s-6)}$	$\frac{-1}{(s-5)^3}$
$\frac{1}{(s-5)^2(s-6)}$	$\frac{-1}{(s-5)^2}$
$\frac{1}{(s-5)(s-6)}$	$\frac{-1}{s-5}$
$\frac{1}{s-6}$	

13. Use the technique of distinct linear factors (Example 5).
$\frac{13/8}{s-5} - \frac{5/8}{s+3}$

15. $\frac{23}{12(s-5)} + \frac{37}{12(s+7)}$

17. $\frac{25}{8(s-7)} - \frac{9}{8(s+1)}$

19. $\frac{1}{2(s+5)} - \frac{1}{2(s-1)} + \frac{1}{s-2}$

21. $\frac{7}{(s+4)^4}$

23. Use Theorem 1 to write

$$\frac{s^2+s-3}{(s+3)^3} = \frac{A_1}{(s+3)^3} + \frac{p_1(s)}{(s+3)^2}$$

$$\text{where} \quad A_1 = \left.\frac{s^2+s-3}{1}\right|_{s=-3} = 3$$

$$\text{and} \quad p_1(s) = \frac{1}{s+3}(s^2+s-3-(3)(1)) = \frac{1}{s+3}(s^2+s-6) = s-2$$

Continuing gives

$$\frac{s-2}{(s+3)^2} = \frac{A_2}{(s+3)^2} + \frac{p_2(s)}{s+3}$$

$$\text{where } A_2 = \left.\frac{s-2}{1}\right|_{s=-3} = -5$$

$$\text{and } p_2(s) = \frac{1}{s+3}(s-2-(-5)(1)) = \frac{1}{s+3}(s+3) = 1$$

Thus $\dfrac{s^2+s-3}{(s+3)^3} = \dfrac{3}{(s+3)^3} - \dfrac{5}{(s+3)^2} + \dfrac{1}{s+3}$

Alternate Solution: Write $s = (s+3) - 3$ so that

$$\begin{aligned}\frac{s^2+s-3}{(s+3)^3} &= \frac{((s+3)-3)^2 + ((s+3)-3) - 3}{(s-3)^3} \\ &= \frac{(s+3)^2 - 5(s+3) + 3}{(s+3)^3} \\ &= \frac{3}{(s+3)^3} - \frac{5}{(s+3)^2} + \frac{1}{s+3}.\end{aligned}$$

25. $\dfrac{s^2-6s+7}{(s^2-4s-5)^2} = \dfrac{s^2-6s+7}{(s+1)^2(s-5)^2}$, so use Theorem 1 to compute the $(s+1)$-chain:

$$\frac{s^2-6s+7}{(s+1)^2(s-5)^2} = \frac{A_1}{(s+1)^2} + \frac{p_1(s)}{(s+1)(s-5)^2}$$

$$\begin{aligned}\text{where } A_1 &= \left.\frac{s^2-6s+7}{(s-5)^2}\right|_{s=-1} = \frac{7}{18} \\ \text{and } p_1(s) &= \frac{1}{s+1}(s^2-6s+7-(7/18)(s-5)^2) \\ &= \frac{1}{s+1}(11s^2-38s-49)/18 = \frac{1}{18}(11s-49)\end{aligned}$$

Continuing gives

$$\frac{1}{18}\frac{11s-49}{(s+1)(s-5)^2} = \frac{A_2}{s+1} + \frac{p_2(s)}{(s-5)^2}$$

$$\begin{aligned}\text{where } A_2 &= \left.\frac{1}{18}\frac{11s-49}{(s-5)^2}\right|_{s=-1} = -5/54 \\ \text{and } p_2(s) &= \frac{1}{s+1}((11s-49)/18-(-5/54)(s-5)^2) = (5s-22)/54\end{aligned}$$

Thus $\dfrac{s^2-6s+7}{(s+1)^2(s-5)^2} = \dfrac{1/18}{(s+1)^2} - \dfrac{5/54}{s+1} + \dfrac{(5s-22)/54}{(s-5)^2}$ Now either continue with Theorem 1 or replace s with $s=(s-5)+5$ in the numerator of the last fraction to finish the calculation and get $\dfrac{s^2-6s+7}{(s+1)^2(s-5)^2} = \dfrac{1}{54}\left(\dfrac{5}{s-5} + \dfrac{21}{(s+1)^2} + \dfrac{3}{(s-5)^2} - \dfrac{5}{s+1}\right)$

27. Use Theorem 1 to compute the $(s+2)$-chain:

$$\begin{aligned}
\frac{s}{(s+2)^2(s+1)^2} &= \frac{A_1}{(s+2)^2} + \frac{p_1(s)}{(s+2)(s+1)^2} \\
\text{where} \quad A_1 &= \left.\frac{s}{(s+1)^2}\right|_{s=-2} = -2 \\
\text{and} \quad p_1(s) &= \frac{1}{s+2}(s-(-2)(s+1)^2) \\
&= \frac{2s^2+5s+2}{s+2} = \frac{(2s+1)(s+1)}{s+2} = 2s+1
\end{aligned}$$

Continuing gives

$$\begin{aligned}
\frac{2s+1}{(s+2)(s+1)^2} &= \frac{A_2}{s+2} + \frac{p_2(s)}{(s+1)^2} \\
\text{where} \quad A_2 &= \left.\frac{2s+1}{(s+1)^2}\right|_{s=-2} = -3 \\
\text{and} \quad p_2(s) &= \frac{1}{s+2}(2s+1-(-3)(s+1)^2) = 3s+2
\end{aligned}$$

Thus $\dfrac{s}{(s+2)^2(s+1)^2} = \dfrac{-2}{(s+2)^2} - \dfrac{3}{s+2} + \dfrac{3s+2}{(s+1)^2}$. Now continue using Theorem 1 or replace s by $(s+1)-1$ in the numerator of the last fraction to get $\dfrac{s}{(s+2)^2(s+1)^2} = \dfrac{-2}{(s+2)^2} - \dfrac{3}{s+2} - \dfrac{1}{(s+1)^2} + \dfrac{3}{s+1}$

29. Use Theorem 1 to compute the $(s-3)$-chain:

$$\begin{aligned}
\frac{8s}{(s-1)(s-2)(s-3)^3} &= \frac{A_1}{(s-3)^3} + \frac{p_1(s)}{(s-1)(s-2)(s-3)^2} \\
\text{where} \quad A_1 &= \left.\frac{8s}{(s-1)(s-2)}\right|_{s=3} = 12 \\
\text{and} \quad p_1(s) &= \frac{1}{s-3}(8s-(12)(s-1)(s-2)) \\
&= \frac{-12s^2+44s-24}{s-3} = \frac{(-12s+8)(s-3)}{s-3} = -12s+8
\end{aligned}$$

For the second step in the $(s-3)$-chain:

$$\begin{aligned}
\frac{-12s+8}{(s-1)(s-2)(s-3)^2} &= \frac{A_2}{(s-3)^2} + \frac{p_2(s)}{(s-1)(s-2)(s-3)^2} \\
\text{where} \quad A_2 &= \left.\frac{-12s+8}{(s-1)(s-2)}\right|_{s=3} = -14 \\
\text{and} \quad p_2(s) &= \frac{1}{s-3}(-12s+8-(-14)(s-1)(s-2)) \\
&= \frac{14s^2-54s+36}{s-3} = \frac{(14s-12)(s-3)}{s-3} = 14s-12
\end{aligned}$$

Continuing gives

$$\begin{aligned}
\frac{14s-12}{(s-1)(s-2)(s-3)^2} &= \frac{A_3}{s-3} + \frac{p_3(s)}{(s-1)(s-2)} \\
\text{where} \quad A_3 &= \left.\frac{14s-12}{(s-1)(s-2)}\right|_{s=3} = 15 \\
\text{and} \quad p_3(s) &= \frac{1}{s-3}(14s-12-(15)(s-1)(s-2)) = -15s+14
\end{aligned}$$

Thus $\frac{8s}{(s-1)(s-3)(s-3)^3} = \frac{12}{(s-3)^3} - \frac{14}{(s-3)^2} + \frac{15}{s-3} + \frac{-15s+14}{(s-1)(s-2)}$. The last fraction has a denominator with distinct linear factors so we get $\frac{8s}{(s-1)(s-3)(s-3)^3} = \frac{12}{(s-3)^3} + \frac{-14}{(s-3)^2} + \frac{15}{s-3} + \frac{-16}{s-2} + \frac{1}{s-1}$

31. Use Theorem 1 to compute the $(s-2)$-chain:

$$\begin{aligned}
\frac{s}{(s-2)^2(s-3)^2} &= \frac{A_1}{(s-2)^2} + \frac{p_1(s)}{(s-2)(s-3)^2} \\
\text{where} \quad A_1 &= \left.\frac{s}{(s-3)^2}\right|_{s=2} = 2 \\
\text{and} \quad p_1(s) &= \frac{1}{s-2}(s-(2)(s-3)^2) \\
&= \frac{-2s^2+13s-18}{s-2} = \frac{(-2s+9)(s-2)}{s-2} = -2s+9
\end{aligned}$$

Continuing gives

$$\begin{aligned}
\frac{-2s+9}{(s-2)(s-3)^2} &= \frac{A_2}{s-3} + \frac{p_2(s)}{(s-3)^2} \\
\text{where} \quad A_2 &= \left.\frac{-2s+9}{(s-3)^2}\right|_{s=2} = 5 \\
\text{and} \quad p_2(s) &= \frac{1}{s-2}(-2s+9-(5)(s-3)^2) = -5s+18
\end{aligned}$$

Thus $\frac{s}{(s-2)^2(s-3)^2} = \frac{2}{(s-2)^2} + \frac{5}{s-2} + \frac{-5s+18}{(s-3)^2}$. Now continue using Theorem 1 or replace s by $(s-3)+3$ in the numerator of the last fraction to get $\frac{s}{(s-2)^2(s-3)^2} = \frac{2}{(s-2)^2} + \frac{5}{s-2} + \frac{3}{(s-3)^2} - \frac{5}{s-3}$

33. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}
\mathcal{L}\{y''+2y'+y\} &= \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} \\
&= s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) \\
&= (s^2+2s+1)Y(s).
\end{aligned}$$

Since $\mathcal{L}\left\{9e^{2t}\right\} = \frac{9}{s-2}$ we get

$$Y(s) = \frac{9}{(s+1)^2(s-2)}.$$

A partial fraction decomposition gives

The $(s+1)$ **-chain**	
$\dfrac{9}{(s+1)^2(s-2)}$	$\dfrac{-3}{(s+1)^2}$
$\dfrac{3}{(s+1)(s-2)}$	$\dfrac{-1}{s+1}$
$\dfrac{1}{(s-2)}$	

It follows that

$$Y(s) = \frac{-3}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s-2}$$

and

$$y(t) = -3te^{-t} - e^{-t} + e^{2t}.$$

35. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}
\mathcal{L}\{y'' - 4y' - 5y\} &= \mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} \\
&= s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) - 5Y(s) \\
&= (s^2 - 4s - 5)Y(s) + s - 5.
\end{aligned}$$

Since $\mathcal{L}\{150t\} = 150/s^2$ we get the algebraic equation

$$(s^2 - 4s - 5)Y(s) + s - 5 = \frac{150}{s^2}.$$

Hence,

$$\begin{aligned}
Y(s) &= \frac{-s+5}{(s+1)(s-5)} + \frac{150}{s^2(s+1)(s-5)} \\
&= \frac{-1}{s+1} + \frac{150}{s^2(s+1)(s-5)}.
\end{aligned}$$

For the second term we start with the s-chain to get the following partial fraction decomposition

The s -chain	
$\dfrac{150}{s^2(s+1)(s-5)}$	$\dfrac{-30}{s^2}$
$\dfrac{30(s-4)}{s(s+1)(s-5)}$	$\dfrac{24}{s}$
$\dfrac{-244s+124}{(s+1)(s-5)}$	$\dfrac{-25}{s+1}$
$\dfrac{1}{s-5}$	

Thus

$$Y(s) = \frac{-30}{s^2} + \frac{24}{s} - \frac{26}{s+1} + \frac{1}{s-5}$$

and Table 2.2 gives $y(t) = -30t + 24 - 26e^{-t} + e^{5t}$.

37. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned} \mathcal{L}\{y'' - 3y' + 2y\} &= \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} \\ &= s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) \\ &= (s^2 - 3s + 2)Y(s) - 2s + 3. \end{aligned}$$

Since $\mathcal{L}\{4\} = 4/s$ we get the algebraic equation

$$(s-1)(s-2)Y(s) - 2s + 3 = \frac{4}{s}.$$

Hence,

$$Y(s) = \frac{2s-3}{(s-1)(s-2)} + \frac{4}{s(s-1)(s-2)}.$$

Each term has denominator a product of distinct linear terms. It is easy to see that

$$\frac{2s-3}{(s-1)(s-2)} = \frac{1}{s-1} + \frac{1}{s-2}$$

and

$$\frac{4}{s(s-1)(s-2)} = \frac{2}{s} - \frac{4}{s-1} + \frac{2}{s-2}.$$

Thus

$$Y(s) = \frac{2}{s} + \frac{3}{s-2} - \frac{3}{s-1}$$

and Table 2.2 gives $y(t) = 2 + 3e^{2t} - 3e^{t}$.

Section 2.4

1. Note that $s = i$ is a root of s^2+1. Applying Theorem 1 gives

$$\begin{aligned}
\frac{1}{(s^2+1)^2(s^2+2)} &= \frac{B_1 s + C_1}{(s^2+1)^2} + \frac{p_1(s)}{(s^2+1)(s^2+2)} \\
\text{where} \quad B_1 i + C_1 &= \left.\frac{1}{(s^2+2)}\right|_{s=i} = \frac{1}{i^2+2} = 1 \\
&\Rightarrow B_1 = 0 \quad \text{and } C_1 = 1 \\
\text{and} \quad p_1(s) &= \frac{1}{s^2+1}(1-(1)(s^2+2)) \\
&= \frac{-s^2-1}{s^2+1} = -1.
\end{aligned}$$

We now apply Theorem 1 on the remainder term $\frac{-1}{(s^2+1)(s^2+2)}$.

$$\begin{aligned}
\frac{-1}{(s^2+1)(s^2+2)} &= \frac{B_2 s + C_2}{(s^2+1)} + \frac{p_2(s)}{(s^2+2)} \\
\text{where} \quad B_2 i + C_2 &= \left.\frac{-1}{(s^2+2)}\right|_{s=i} = -1 \\
&\Rightarrow B_2 = 0 \text{ and } C_2 = -1 \\
\text{and} \quad p_2(s) &= \frac{1}{s^2+1}(-1-(-1)(s^2+2)) \\
&= \frac{s^2+1}{s^2+1} = 1.
\end{aligned}$$

Thus the (s^2+1)-chain is

The s^2+1 -chain	
$\frac{1}{(s^2+1)^2(s^2+2)}$	$\frac{1}{(s^2+1)^2}$
$\frac{-1}{(s^2+1)(s^2+2)}$	$\frac{-1}{(s^2+1)}$
$\frac{1}{s^2+2}$	

3. Note that $s = \sqrt{3}i$ is a root of s^2+3. Applying Theorem 1 gives

$$\begin{aligned}
\frac{8s+8s^2}{(s^2+3)^3(s^2+1)} &= \frac{B_1 s + C_1}{(s^2+3)^3} + \frac{p_1(s)}{(s^2+3)^2(s^2+1)} \\
\text{where} \quad B_1\sqrt{3}i + C_1 &= \left.\frac{8s+8s^2}{(s^2+1)}\right|_{s=\sqrt{3}i} = \frac{8\sqrt{3}i + 8(\sqrt{3}i)^2}{(\sqrt{3}i)^2+1} \\
&= -4\sqrt{3}i + 12 \\
&\Rightarrow B_1 = -4 \quad \text{and } C_1 = 12 \\
\text{and} \quad p_1(s) &= \frac{1}{s^2+3}(8s+8s^2-(-4s+12)(s^2+1)) \\
&= \frac{4s^3-4s^2+12s-12}{s^2+3} = 4(s-1).
\end{aligned}$$

Apply Theorem 1 a second time on the remainder term $\frac{4s-4}{(s^2+3)^2(s^2+1)}$.

$$\begin{aligned}
\frac{4s-4}{(s^2+3)^2(s^2+1)} &= \frac{B_2 s + C_2}{(s^2+3)^2} + \frac{p_2(s)}{(s^2+3)(s^2+1)} \\
\text{where} \quad B_2\sqrt{3}i + C_2 &= \left.\frac{4s-4}{(s^2+1)}\right|_{s=\sqrt{3}i} = -2\sqrt{3}i + 2 \\
&\Rightarrow B_2 = -2 \text{ and } C_2 = 2 \\
\text{and} \quad p_2(s) &= \frac{1}{s^2+3}(4s-4-(-2s+2)(s^2+1)) \\
&= \frac{2s^3-2s^2+6s-6}{s^2+3} = 2s-2.
\end{aligned}$$

A third application of Theorem 1 on the remainder term $\frac{2s-2}{(s^2+3)(s^2+1)}$ gives

$$\begin{aligned}
\frac{2s-2}{(s^2+3)(s^2+1)} &= \frac{B_3 s + C_3}{(s^2+3)} + \frac{p_3(s)}{(s^2+1)} \\
\text{where} \quad B_3\sqrt{3}i + C_3 &= \left.\frac{2s-2}{(s^2+1)}\right|_{s=\sqrt{3}i} = -\sqrt{3}i + 1 \\
&\Rightarrow B_3 = -1 \text{ and } C_3 = 1 \\
\text{and} \quad p_3(s) &= \frac{1}{s^2+3}(2s-2-(-s+1)(s^2+1)) \\
&= \frac{s^3-s^2+3s-3}{s^2+3} = s-1.
\end{aligned}$$

Thus the (s^2+3)-chain is

The s^2+3 -chain	
$\dfrac{8s+8s^2}{(s+3)^3(s^2+1)}$	$\dfrac{12-4s}{(s^2+3)^3}$
$\dfrac{4(s-1)}{(s^2+3)^2(s^2+1)}$	$\dfrac{2-2s}{(s^2+3)^2}$
$\dfrac{2(s-1)}{(s^2+3)(s^2+1)}$	$\dfrac{1-s}{s^2+3}$
$\dfrac{s-1}{s^2+1}$	

5. Note that $s^2+2s+2=(s+1)^2+1$ so $s=-1\pm i$ are the roots of s^2+2s+2. We will use the root $s=-1+i$ for the partial fraction algorithm. Applying Theorem 1 gives

$$\frac{1}{(s^2+2s+2)^2(s^2+2s+3)^2} = \frac{B_1s+C_1}{(s^2+2s+2)^2} + \frac{p_1(s)}{(s^2+2s+2)(s^2+2s+3)^2}$$

$$\begin{aligned}
\text{where}\quad B_1(-1+i)+C_1 &= \left.\frac{1}{(s^2+2s+3)^2}\right|_{s=-1+i} \\
&= \frac{1}{((-1+i)^2+2)^2}=1 \\
&\Rightarrow B_1=0 \quad \text{and } C_1=1 \\
\text{and}\quad p_1(s) &= \frac{1-(1)(s^2+2s+3)^2}{s^2+2s+2} \\
&= \frac{-(s^2+2s+4)(s^2+2s+2)}{s^2+2s+2} \\
&= -(s^2+2s+4).
\end{aligned}$$

Now apply Theorem 1 to the remainder term $\dfrac{-(s^2+2s+4)}{(s^2+2s+2)(s^2+2s+3)^2}$.

$$\begin{aligned}
\frac{-(s^2+2s+4)}{(s^2+2s+2)(s^2+2s+3)^2} &= \frac{B_2 s + C_2}{(s^2+2s+2)} + \frac{p_2(s)}{(s^2+2s+3)^2} \\
\text{where} \quad B_2(-1+i)+C_2 &= \left.\frac{-(s^2+2s+4)}{(s^2+2s+3)}\right|_{s=-1+i} = -2 \\
&\Rightarrow \quad B_2 = 0 \text{ and } C_2 = -2 \\
\text{and} \quad p_2(s) &= \frac{-(s^2+2s+4)-(-2)(s^2+2s+3)^2}{s^2+2s+2} \\
&= \frac{(2(s+1)^2+5)((s+1)^2+1)}{s^2+2s+2} \\
&= 2s^2+4s+7.
\end{aligned}$$

Thus the (s^2+2s+2)-chain is

The s^2+2s+2 **-chain**	
$\frac{1}{(s^2+2s+2)^2(s^2+2s+3)^2}$	$\frac{1}{(s^2+2s+2)^2}$
$\frac{-(s^2+2s+4)}{(s^2+2s+2)(s^2+2s+3)^2}$	$\frac{-2}{s^2+2s+2}$
$\frac{2s^2+4s+7}{(s^2+2s+3)^2}$	

7. Use Theorem 1 of Section 2.3 to compute the $(s-3)$-chain:

$$\frac{s}{(s^2+1)(s-3)} = \frac{A_1}{s-3} + \frac{p_1(s)}{s^2+1}$$

$$\begin{aligned}
\text{where} \quad A_1 &= \left.\frac{s}{s^2+1}\right|_{s=3} = \frac{3}{10} \\
\text{and} \quad p_1(s) &= \frac{1}{s-3}(s-(3/10)(s^2+1)) = \frac{1}{10(s-3)}(-3s^2+10s-3) \\
&= \frac{-3s+1}{10}
\end{aligned}$$

Since the remainder term $\frac{-3s+1}{10(s^2+1)}$ is already a simple partial fraction, we conclude

$$\frac{s}{(s^2+1)(s-3)} = \frac{1}{10}\left(\frac{3}{s-3} + \frac{1-3s}{s^2+1}\right)$$

9. We compute the (s^2+4)-chain:

$$\frac{9s^2}{(s^2+4)^2(s^2+1)} = \frac{B_1 s + C_1}{(s^2+4)^2} + \frac{p_1(s)}{(s^2+4)(s^2+1)}$$

$$\begin{aligned} \text{where} \quad B_1(2i)+C_1 &= \left.\frac{9s^2}{s^2+1}\right|_{s=2i} = 12 \\ &\Rightarrow B_1 = 0 \quad \text{and } C_1 = 12 \\ \text{and} \quad p_1(s) &= \frac{1}{s^2+4}(9s^2 - 12(s^2+1)) = \frac{-3(s^2+4)}{s^2+4} \\ &= -3 \end{aligned}$$

Now compute the next term in s^2+4-chain.

$$\frac{-3}{(s^2+4)(s^2+1)} = \frac{B_2 s + C_2}{s^2+4} + \frac{p_2(s)}{s^2+1}$$

$$\begin{aligned} \text{where} \quad B_2(2i)+C_1 &= \left.\frac{-3}{s^2+1}\right|_{s=2i} = 1 \\ &\Rightarrow B_2 = 0 \quad \text{and } C_2 = 1 \\ \text{and} \quad p_2(s) &= \frac{1}{s^2+4}(-3-(s^2+1)) \\ &= \frac{-(s^2+1)}{s^2+1} = -1. \end{aligned}$$

Since the remainder term $\frac{-1}{s^2+1}$ is a simple partial fraction, we conclude that the complete partial fraction decomposition is

$$\frac{9s^2}{(s^2+4)^2(s^2+1)} = \frac{12}{(s^2+4)^2} + \frac{1}{s^2+4} - \frac{1}{s+1}$$

11. Use Theorem of Section 2.3 1 to compute the $(s-3)$-chain:

$$\frac{2}{(s^2-6s+10)(s-3)} = \frac{A_1}{s-3} + \frac{p_1(s)}{(s^2-6s+10)}$$

$$\begin{aligned} \text{where} \quad A_1 &= \left.\frac{2}{(s^2-6s+10)}\right|_{s=3} = 2 \\ \text{and} \quad p_1(s) &= \frac{1}{s-3}(2-(2)(s^2-6s+10)) = \frac{-2s^2+12s-18}{s-3} \\ &= -2s+6 \end{aligned}$$

Since the remainder term $\frac{-2s+6}{s^2-6s+10}$ is a simple partial fraction, we conclude that the complete partial fraction decomposition is $\frac{2}{(s^2-6s+10)(s-3)} = \frac{2}{s-3} + \frac{6-2s}{(s-3)^2+1}$

13. Note that $s^2-4s+8 = (s-2)^2+2$ so $s = 2\pm 2i$ are the roots of s^2-4s+8. We will use the root $s = 2+2i$ to compute the (s^2-4s+8)-chain. Applying Theorem 1 gives

$$\begin{aligned}
\frac{25}{(s^2-4s+8)^2(s-1)} &= \frac{B_1 s + C_1}{(s^2-4s+8)^2} \\
&\quad + \frac{p_1(s)}{(s^2-4s+8)(s-1)}
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad B_1(2+2i)+C_1 &= \left.\frac{25}{s-1)}\right|_{s=2+2i} \\
&= \frac{25}{2i+1} = 5-10i \\
&\Rightarrow B_1 = -5 \quad \text{and } C_1 = 15 \\
\text{and} \quad p_1(s) &= \frac{25-(-5s+15)(s-1)}{s^2-4s+8} \\
&= \frac{(5)(s^2-4s+8)}{s^2-4s+8} \\
&= 5.
\end{aligned}$$

Now apply Theorem 1 to the remainder term $\frac{5}{(s^2-4s+8)(s-1)}$.

$$\frac{5}{(s^2-4s+8)(s-1)} = \frac{B_2 s + C_2}{(s^2-4s+8)} + \frac{p_2(s)}{s-1}$$

$$\begin{aligned}
\text{where} \quad B_2(2+2i)+C_2 &= \left.\frac{5)}{s-1}\right|_{s=2+2i} = 1-2i \\
&\Rightarrow B_2 = -1 \text{ and } C_2 = 3 \\
\text{and} \quad p_2(s) &= \frac{5-(3-s)(s-1)}{s^2-4s+8} \\
&= \frac{(1)(s^2-4s+8)}{s^2-4s+8} \\
&= 1.
\end{aligned}$$

Thus the partial fraction expansion is $\frac{25}{(s^2-4s+8)^2(s-1)} = \frac{-5s+15}{(s^2-4s+8)^2} + \frac{-s+3}{s^2-4s+8} + \frac{1}{s-1}$

15. Note that $s^2+4s+5 = (s+2)^2+1$ so $s = -2 \pm i$ are the roots of s^2+4s+5. We will use the root $s = -2+i$ to compute the (s^2+4s+5)-chain. Applying Theorem 1 gives

$$\begin{aligned}\frac{s+1}{(s^2+4s+5)^2(s^2+4s+6)^2} &= \frac{B_1s+C_1}{(s^2+4s+5)^2} \\ &\quad + \frac{p_1(s)}{(s^2+4s+5)(s^2+4s+6)^2}\end{aligned}$$

$$\begin{aligned}\text{where}\quad B_1(-2+i)+C_1 &= \left.\frac{s+1}{(s^2+4s+6)^2)}\right|_{s=-2+i} \\ &= -1+i \\ &\Rightarrow B_1 = 1 \quad \text{and } C_1 = 1 \\ \text{and}\quad p_1(s) &= \frac{s+1-(s+1)(s^2+4s+6)^2}{s^2+4s+5} \\ &= \frac{-(s+1)((s^2+4s+6)^2-1)}{s^2+4s+5} \\ &= \frac{-(s+1)(s^2+4s+7)(s^2+4s+5)}{s^2+4s+5} \\ &= -(s+1)(s^2+4s+7).\end{aligned}$$

Now apply Theorem 1 to the remainder term $\frac{-(s+1)(s^2+4s+7)}{(s^2+4s+5)(s^2+4s+6)^2)}$.

$$\frac{-(s+1)(s^2+4s+7)}{(s^2+4s+5)(s^2+4s+6)^2} = \frac{B_2s+C_2}{(s^2+4s+5)} + \frac{p_2(s)}{(s^2+4s+6)^2}$$

$$\begin{aligned}\text{where}\quad B_2(-2+i)+C_2 &= \left.\frac{-(s+1)(s^2+4s+7)}{(s^2+4s+6)^2}\right|_{s=-2+i} = 2-2i \\ &\Rightarrow B_2 = -2 \text{ and } C_2 = -2 \\ \text{and}\quad p_2(s) &= \frac{-(s+1)(s^2+4s+7)-(-2s-2)(s^2+4s+6)^2}{s^2+4s+5} \\ &= \frac{(s+1)(2(s^2+4s+6)+1)(s^2+4s+5)}{s^2+4s+5} \\ &= (s+1)(2(s^2+4s+6)+1).\end{aligned}$$

The remainder term is

$$\frac{(s+1)(2(s^2+4s+6)+1)}{(s^2+4s+6)^2} = \frac{s+1}{(s^2+4s+6)^2} + \frac{2s+2}{s^2+4s+6}$$

so the partial fraction expansion of the entire rational function is

$$\begin{aligned}\frac{s+1}{(s^2+4s+5)^2(s^2+4s+6)^2} &= \frac{s+1}{(s^2+4s+6)^2} + \frac{2s+2}{s^2+4s+6} \\ &\quad + \frac{s+1}{(s^2+4s+5)^2} - \frac{2s+2}{s^2+4s+5}\end{aligned}$$

17. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}\mathcal{L}\{y''+4y'+4y\} &= \mathcal{L}\{y''\}+4\mathcal{L}\{y'\}+4\mathcal{L}\{y\} \\ &= s^2Y(s)-sy(0)-y'(0)+4(sY(s)-y(0))+4Y(s) \\ &= (s^2+4s+4)Y(s)-1.\end{aligned}$$

Since $\mathcal{L}\{4\cos 2t\} = 4s/(s^2+4)$ we get the algebraic equation

$$(s+4)^2Y(s)-1 = \frac{4s}{s^2+4}.$$

Hence,

$$Y(s) = \frac{1}{(s+2)^2} + \frac{4s}{(s^2+4)(s+2)^2}.$$

The (s^2+4)-chain for the second term is

The (s^2+4)**-chain**	
$\frac{4s}{(s^2+4)(s+2)^2}$	$\frac{1}{s^2+4}$
$\frac{-1}{(s+2)^2}$	

Thus

$$Y(s) = \frac{1}{s^2+4}$$

and Table 2.2 gives $y(t) = \frac{1}{2}\sin 2t$

19. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}\mathcal{L}\{y''+4y\} &= \mathcal{L}\{y''\}+4\mathcal{L}\{y\}\\ &= s^2Y(s)-sy(0)-y'(0)+4Y(s)\\ &= (s^2+4)Y(s)-1.\end{aligned}$$

Since $\mathcal{L}\{\sin 3t\} = 3/(s^2+9)$ we get the algebraic equation

$$(s^2+4)Y(s)-1=\frac{3}{s^2+9}.$$

Hence,

$$Y(s)=\frac{1}{s^2+4}+\frac{3}{(s^2+9)(s^2+4)}.$$

Using quadratic partial fraction recursion we obtain the (s^2+9)-chain

The (s^2+9)**-chain**	
$\frac{3}{(s^2+9)(s^2+4)}$	$\frac{-3/5}{s^2+9}$
$\frac{3/5}{s^2+4}$	

Thus

$$Y(s)=\frac{8}{5}\frac{1}{s^2+4}-\frac{3}{5}\frac{1}{s^2+9}=\frac{4}{5}\frac{2}{s^2+4}-\frac{1}{5}\frac{3}{s^2+9}$$

and Table 2.2 gives $y(t)=\frac{4}{5}\sin 2t-\frac{1}{5}\sin 3t$

Section 2.5

1. $\mathcal{L}^{-1}\{-5/s\}=-5\mathcal{L}^{-1}\{1/s\}=-5$

3. $\mathcal{L}^{-1}\left\{\frac{3}{s^2}-\frac{4}{s^3}\right\}=3\mathcal{L}^{-1}\left\{1/s^2\right\}-2\mathcal{L}^{-1}\left\{2/s^3\right\}=3t-2t^2$

5. $\mathcal{L}^{-1}\left\{\frac{3s}{s^2+4}\right\}=3\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}=3\cos 2t$

7. First, we have $s^2+6s+9=(s+3)^2$. Partial fractions gives $\dfrac{2s-5}{s^2+6s+9}=\dfrac{-11}{(s+3)^2}+\dfrac{2}{s+3}$. So $\mathcal{L}^{-1}\left\{\dfrac{2s-5}{(s+3)^2}\right\}=-11te^{-3t}+2e^{-3t}$

9. $\dfrac{6}{s^2+2s-8}=\dfrac{6}{(s-2)(s+4)}=\dfrac{-1}{s+4}+\dfrac{1}{s-2}$. So $\mathcal{L}^{-1}\left\{\dfrac{6}{s^2+2s-8}\right\}=e^{2t}-e^{-4t}$

11. $\dfrac{2s^2-5s+1}{(s-2)^4}=\dfrac{-1}{(s-2)^4}+\dfrac{3}{(s-2)^3}+\dfrac{2}{(s-2)^2}$. So $\mathcal{L}^{-1}\left\{\dfrac{2s^2-5s+1}{(s-2)^4}\right\}=\dfrac{-1}{6}t^3e^{2t}+\dfrac{3}{2}t^2e^{2t}+2te^{2t}$

13. $\dfrac{4s^2}{(s-1)^2(s+1)^2}=\dfrac{1}{(s-1)^2}+\dfrac{1}{s-1}+\dfrac{1}{(s+1)^2}-\dfrac{1}{s+1}$. So $\mathcal{L}^{-1}\left\{\dfrac{4s^2}{(s-1)^2(s+1)^2}\right\}=te^t+e^t+te^{-t}-e^{-t}$

15. $\dfrac{8s+16}{(s^2+4)(s-2)^2}=\dfrac{4}{(s-2)^2}-\dfrac{1}{s-2}+\dfrac{s}{s^2+4}-\dfrac{2}{s^2+4}$. So $\mathcal{L}^{-1}\left\{\dfrac{8s+16}{(s^2+4)(s-2)^2}\right\}=4te^{2t}-e^{2t}+\cos 2t-\sin 2t$

17. $\dfrac{12}{s^2(s+1)(s-2)}=\dfrac{-6}{s^2}+\dfrac{3}{s}-\dfrac{4}{s+1}+\dfrac{1}{s-2}$. So $\mathcal{L}^{-1}\left\{\dfrac{12}{s^2(s+1)(s-2)}\right\}=3-6t+e^{2t}-4e^{-t}$

19. First we have $s^2+2s+5=(s+1)^2+4$. So $\dfrac{2s}{s^2+2s+5}=\dfrac{2s}{(s+1)^2+4}=\dfrac{2(s+1)-2}{(s+1)^2+4}=2\dfrac{s+1}{(s+1)^2+4}-\dfrac{2}{(s+1)^2+4}$. The First Translation principle gives $\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2+2s+5}\right\}=2\mathcal{L}^{-1}\left\{\dfrac{s+1}{(s+1)^2+4}\right\}-\mathcal{L}^{-1}\left\{\dfrac{2}{(s+1)^2+4}\right\}=2e^{-t}\cos 2t-e^{-t}\sin 2t$

21. $\dfrac{s-1}{s^2-8s+17}=\dfrac{s-4}{(s-4)^2+1}+3\dfrac{1}{(s-4)^2+1}$. Thus $\mathcal{L}^{-1}\left\{\dfrac{s-1}{s^2-8s+17}\right\}=e^{4t}\cos t+3e^{4t}\sin t$

23. $\dfrac{s-1}{s^2-2s+10}=\dfrac{s-1}{(s-1)^2+3^2}$. Thus $\mathcal{L}^{-1}\left\{\dfrac{s-1}{s^2-2s+10}\right\}=e^t\cos 3t$

25. $\mathcal{L}^{-1}\left\{\dfrac{8s}{(s^2+4)^2}\right\}=8\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+2^2)^2}\right\}=\dfrac{8}{2\cdot 2^2}(2t\sin 2t)=2t\sin 2t$

27. We first complete the square $s^2+4s+5=(s+2)^2+1$. By the translation principle we get $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s^2+4s+5)^2}\right\}=2\mathcal{L}^{-1}\left\{\dfrac{(s+2)-2}{((s+2)^2+1)^2}\right\}=$

$2e^{-2t}\left(\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}\right)=2e^{-2t}\left(\frac{1}{2}t\sin t-2(\frac{1}{2}(\sin t-t\cos t)\right)=$ $2te^{-2t}\cos t+(t-2)e^{-2t}\sin t$

29. We first complete the square $s^2+8s+17=(s+4)^2+1$. By the translation principle we get $\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+8s+17)^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{(s+4)-4}{((s+4)^2+1)^2}\right\} =$ $2e^{-4t}\left(\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}-4\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}\right)=2e^{-4t}\left(\frac{1}{2}t\sin t-4(\frac{1}{2}(\sin t-t\cos t)\right)=$ $4te^{-4t}\cos t+(t-4)e^{-4t}\sin t$

31. We first complete the square $s^2-2s+5=(s-1)^2+2^2$. By the translation principle we get $\mathcal{L}^{-1}\left\{\frac{1}{(s^2-2s+5)^3}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{((s-1)^2+2^2)^3}\right\}=$
$e^t\left(\mathcal{L}^{-1}\left\{\frac{1}{(s^2+2^2)^3}\right\}\right)=e^t\frac{1}{2}\left(\mathcal{L}^{-1}\left\{\frac{2}{(s^2+2^2)^3}\right\}\right)$
$=e^t\frac{1}{2\cdot 8\cdot 2^4}\left((3-(2t)^2)\sin 2t-6t\cos 2t\right)$
$=\frac{1}{256}\left((3-4t^2)e^t\sin 2t-6te^t\cos 2t\right)$

33. We first complete the square $s^2-8s+17=(s-4)^2+1$. By the translation principle we get $\mathcal{L}^{-1}\left\{\frac{s-4}{(s^2-8s+17)^4}\right\}=\mathcal{L}^{-1}\left\{\frac{s-4}{((s-4)^2+1)^4}\right\}=$
$e^{4t}\left(\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^4}\right\}\right)=e^{4t}\frac{1}{48}\left((3t-t^3)\sin t-3t^2\cos t\right)$
$=\frac{1}{48}\left((-t^3+3t)e^{4t}\cos t-3t^2e^{4t}\cos t\right)$

35. Apply the Laplace transform to get

$$s^2Y(s)-s+1+Y(s)=\frac{4}{s^2+1}.$$

Solving for $Y(s)$ we get

$$Y(s)=\frac{s-1}{s^2+1}+\frac{4}{(s^2+1)^2}.$$

We use Table 2.5 to get

$$y(t)=\cos t-\sin t+2(\sin t-t\cos t)=\cos t+\sin t-2t\cos t.$$

37. Apply the Laplace transform to get

$$\begin{aligned}(s^2-3)Y(s) &= 4\left(\frac{s}{s^2+1}\right)'' \\ &= 4\left(\frac{1-s^2}{(s^2+1)^2}\right)' \\ &= \frac{8s(s^2-3)}{(s^2+1)^3}\end{aligned}$$

It follows that $Y(s) = \dfrac{8s}{(s^2+1)^3}$. Table 2.5 now gives

$$y(t) = t\sin t - t^2\cos t.$$

39. Compute the partial fraction $\dfrac{1}{(s-a)(s-b)} = \dfrac{1/(a-b)}{s-a} + \dfrac{1/(b-a)}{s-b}$. Then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/(a-b)}{s-a} + \frac{1/(b-a)}{s-b}\right\} = \frac{e^{at}}{a-b} + \frac{e^{bt}}{b-a}.$$

41. Apply the inverse Laplace transform to the partial fraction expansion

$$\frac{1}{(s-a)(s-b)(s-c)} = \frac{1}{(a-b)(a-c)}\frac{1}{s-a} + \frac{1}{(b-a)(b-c)}\frac{1}{s-b} + \frac{1}{(c-a)(c-b)}\frac{1}{(s-c)}.$$

43. Apply the inverse Laplace transform to the partial fraction expansion

$$\frac{s^2}{(s-a)(s-b)(s-c)} = \frac{a^2}{(a-b)(a-c)}\frac{1}{s-a} + \frac{b^2}{(b-a)(b-c)}\frac{1}{s-b} + \frac{c^2}{(c-a)(c-b)}\frac{1}{(s-c)}.$$

45. This is directly from Table 2.4.

47. This is directly from Table 2.4.

49. Apply the inverse Laplace transform to the partial fraction expansion

$$\frac{s^2}{(s-a)^3} = \frac{((s-a)+a)^2}{(s-a)^3} = \frac{1}{s-a} + \frac{2a}{(s-a)^2} + \frac{a^2}{(s-a)^3}.$$

Section 2.6

1. The root of $q(s)$ is 4 with multiplicity 1. Thus $\mathcal{B}_q = \left\{e^{4t}\right\}$

3. $q(s) = s^2 + 5s = s(s+5)$, hence the roots of $q(s)$ are 0 and -5 each with multiplicity 1. Thus $\mathcal{B}_q = \{1,\, e^{-5t}\}$

5. $q(s) = s^2 - 6s + 9 = (s-3)^2$, hence the root of $q(s)$ is 3 with multiplicity 2. Thus $\mathcal{B}_q = \{e^{3t},\, te^{3t}\}$

7. $q(s) = s^2 - s - 6 = (s-3)(s+2)$, hence the root of $q(s)$ are 3 and -2 each with multiplicity 1. Thus $\mathcal{B}_q = \{e^{3t},\, e^{-2t}\}$

9. $q(s) = 6s^2 - 11s + 4 = (3s-4)(2s-1)$ so the roots are 4/3 and 1/2, each with multiplicity 1. Hence $\mathcal{B}_q = \{e^{t/2},\, e^{4t/3}\}$

11. The quadratic formula gives roots $\dfrac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$. Hence $\mathcal{B}_q = \left\{e^{(2+\sqrt{3})t},\, e^{(2-\sqrt{3})t}\right\}$

13. $q(s) = 4s^2 + 12s + 9 = (2s+3)^2$; so the root is $-3/2$ with multiplicity 2 and hence $\mathcal{B}_q = \{e^{-3t/2},\, te^{-3t/2}\}$

15. $q(s) = 4s^2 + 25 = 4(s^2 + (5/2)^2)$. Therefore $q(s)$ has complex roots $\pm\frac{5}{2}i$. Hence $\mathcal{B}_q = \{\cos(5t/2),\, \sin(5t/2)\}$

17. $q(s) = s^2 - 2s + 5 = s^2 - 2s + 1 + 4 = (s-1)^2 + 2^2$. Therefore $q(s)$ has complex roots $1 \pm 2i$. Hence $\mathcal{B}_q = \{e^t \cos 2t,\, e^t \sin 2t\}$

19. $q(s)$ has root -3 with multiplicity 4. Hence
$\mathcal{B}_q = \{e^{-3t}, te^{-3t}, t^2e^{-3t}, t^3e^{-3t}\}$.

21. $q(s) = (s-1)^3$ has root 1 with multiplicity 3. Hence
$\mathcal{B}_q = \{e^t, te^t, t^2e^t\}$.

23. We complete the square to get $q(s) = ((s+2)^2 + 1)^2$. Thus $q(s)$ has complex root $-2 \pm i$ with multiplicity 2 It follows that
$\mathcal{B}_q = \{e^{-2t}\cos t, e^{-2t}\sin t, te^{-2t}\cos t, te^{-2t}\sin t\}$.

25. The complex roots of $q(s)$ are $\pm i$ with multiplicity 4. Thus $\mathcal{B}_q = \{\cos t,\, \sin t,\, t\cos t,\, t\sin t,\, t^2\cos t,\, t^2\sin t,\, t^3\cos t,\, t^3\sin t\}$

Section 2.7

1. Yes.

3. Yes; $\dfrac{t}{e^t} = te^{-t}$.

5. Yes; $t\sin(4t - \dfrac{\pi}{4}) = t(\dfrac{\sqrt{2}}{2}\sin 4t - \dfrac{\sqrt{2}}{2}\cos 4t)$.

7. No.

9. No; $t^{\frac{1}{2}}$ is not a polynomial.

11. No. $\dfrac{1}{\sin 2t}$ is not in $\mathcal{E}$.

13. $s^4 - 1 = (s^2 - 1)(s^2 + 1) = (s - 1)(s + 1)(s^2 + 1)$; The roots ate 1, -1, and $\pm i$ each with multiplicity 1. Hence $\mathcal{B}_q = \{e^t, e^{-t}, \cos t, \sin t\}$.

15. The roots are 1 with multiplicity 3 and -7 with multiplicity 2. Hence $\mathcal{B}_q = \left\{e^t, te^t, t^2e^t, e^{-7t}, te^{-7t}\right\}$.

17. The roots are -2 with multiplicity 3 and $\pm 2i$ with multiplicity 2. Hence $\mathcal{B}_q = \left\{e^{-2t}, te^{-2t}, t^2e^{-2t}, \cos 2t, \sin 2t, t\cos 2t, t\sin 2t\right\}$.

19. We must gather the roots together to get the correct multiplicity. Thus $q(s) = (s - 2)^2(s + 3)^3$. The roots are 2 with multiplicity 2 and -3 with multiplicity 3. Hence $\mathcal{B}_q = \left\{e^{2t}, te^{2t}, e^{-3t}, te^{-3t}, t^2e^{-3t}\right\}$.

21. By completing the square we may write $q(s) = (s+4)^2((s+3)^2+4)^2$ The roots are -4 with multiplicity 2 and $-3 \pm 2i$ with multiplicity 2. Hence $\mathcal{B}_q = \left\{e^{-4t}, te^{-4t}, e^{-3t}\cos 2t, e^{-3t}\sin 2t, te^{-3t}\cos 2t, te^{-3t}\sin 2t\right\}$.

23. First observe that $s^2+2s+10 = (s+1)^2+3^2$ and hence $q(s) = (s-3)^3((s+1)^2+3^2)^2$. The roots are 3 with multiplicity 3 and $-1\pm 3i$ with multiplicity 2. Thus $\mathcal{B}_q = \left\{e^{3t}, te^{3t}, t^2e^{3t}, e^{-t}\cos 3t, e^{-t}\sin 3t, te^{-t}\cos 3t, te^{-t}\sin 3t\right\}$.

25. $2s^3 - 5s^2 + 4s - 1 = (2s - 1)(s - 1)^2$; hence $\mathcal{B}_q = \left\{e^{t/2}, e^t, te^t\right\}$

27. $s^4+5s^2+6 = (s^2+3)(s^2+2)$; hence $\mathcal{B}_q = \left\{\cos\sqrt{3}t, \sin\sqrt{3}t, \cos\sqrt{2}t, \sin\sqrt{2}t\right\}$

29. $r_1(s) = \dfrac{p_1(s)}{q_1(s)}$ with $\deg p_1(s) < \deg q_1(s)$ and $r_2(s) = \dfrac{p_2(s)}{q_2(s)}$ with $\deg p_2(s) < \deg q_2(s)$. Thus, $r_1(s)r_2(s) = \dfrac{p_1(s)p_2(s)}{q_1(s)q_2(s)}$ and

$$\deg(p_1(s)p_2(s) = \deg p_1(s)+\deg p_2(s) < \deg q_1(s)+\deg q_2(s) = \deg(q_1(s)q_2(s)).$$

31. If $r(s) \in \mathcal{R}$ then $r(s) = \dfrac{p(s)}{q(s)}$ where $\deg p(s) = m < n = \deg q(s)$. Then

$$r'(s) = \frac{q(s)p'(s) - q'(s)p(s)}{(q(s))^2},$$

and $\deg(q(s)p'(s) - q'(s)p(s)) \leq \max(\deg(q(s)q'(s)), \deg(q'(s)p(s))) = \max(n + (m - 1), (n - 1) + m) = n + m - 1 < 2n = \deg(q(s))^2$. Hence $r'(s)$ is a proper rational function.

33. By exercise 32 this is true for $n = 1$. Now apply induction. If n is given and we assume the result is true for derivatives of order $n-1$, then $r^{(n-1)} \in \mathcal{R}_{q^n}$ but not in $\mathcal{R}_{q^{n-1}}$. Another application of exercise 32 then shows that $r^{(n)} = \left(r^{(n-1)}\right)' \in \mathcal{R}_{q^{n+1}}$ but not in $\mathcal{R}_{q^n}$.

35. Observe that $e^{t-t_0} = e^{-t_0}e^t$. So the translate of an exponential function is a multiple of an exponential function. Also, if $f(t)$ is a polynomial so is $f(t-t_0)$. By the addition rule for cos we have $\cos b(t-t_0) = \cos bt \cos bt_0 - \sin bt \sin bt_0$ and similarly for sin. It follows that all these translates remain in $\mathcal{E}$. By Exercise 34 the result follows.

37. By linearity of integration it is enough to show this result for $f_n(t) = t^n e^{at}(c_1 \cos bt + c_2 \sin bt)$, where c_1 and c_2 are scalars. Let $I_n(t) = \int f_n(t)\, dt$. First assume $n = 0$. Then a standard trick using integration by parts twice gives

$$\begin{aligned} I_0(t) :&= \int e^{at}(c_1 \cos bt + c_2 \sin bt)\, dt \\ &= \frac{1}{a^2+b^2}((c_1 a - c_2 b)\cos bt + (c_1 b + c_2 a)\sin bt)e^{at}. \end{aligned}$$

Clearly, I_0 is an exponential polynomial. Observe that I_0 is of the same form as f_0. Now assume $n > 0$. Using integration by parts with $u = t^n$ and $dv = (c_1 \cos bt + c_2 \sin bt)\, dt$ we have $I_n(t) = \int t^n(c_1 \cos bt + c_2 \sin bt)\, dt = t^n I_0(t) - n\int t^{n-1} I_0(t)\, dt$. Since $I_0 \in \mathcal{E}$ so are $t^n I_0$ and $t^{n-1} I_0$. By induction we have $\int t^{n-1} I_0(t)\, dt \in \mathcal{E}$. It now follows that $I_n \in \mathcal{E}$.

39. It is enough to show this for each $f \in \mathcal{B}_q$ since differentiation is linear and $\mathcal{E}_q = \text{Span}\, \mathcal{B}_q$. Suppose $f(t) = t^n e^{at} \cos bt$. Then

$$f'(t) = nt^{n-1}e^{at}\cos bt - bt^n e^{at} \sin bt + at^n e^{at} \cos bt.$$

The derivative $f'(t)$ is a linear combination of the simple exponential polynomials $t^{n-1}e^{at}\cos bt$, $t^n e^{at} \sin bt$, and $t^n e^{at} \cos bt$ each of which are in $\mathcal{B}_q$. Hence $f'(t) \in \mathcal{E}_q$. A similar argument applies to $t^n e^{at} \sin bt$.

41. Observe that $e^{t-t_0} = e^{-t_0}e^t$. So the translate of an exponential function is a multiple of an exponential function. Also, if $p(t)$ is a polynomial of degree n the binomial theorem implies that $p(t-t_0)$ is a polynomial of degree n. By the addition rule for cos we have $\cos b(t-t_0) = \cos bt \cos bt_0 - \sin bt \sin bt_0$ and similarly for sin. Thus if $f(t) = t^n e^{at} \cos bt \in \mathcal{B}_q$ then $f(t-t_0)$ is a linear combination of terms in $\mathcal{B}_q$. Since $\mathcal{E}_q = \text{Span}\, \mathcal{B}_q$ it follows that all translates remain in $\mathcal{E}_q$.

Section 2.8

1.

$$\begin{aligned} t * t &= \int_0^t x(t-x)\,dx \\ &= \int_0^t (tx - x^2)\,dx \\ &= \left(t\frac{x^2}{2} - \frac{x^3}{3} \right)\Big|_{x=0}^{x=t} \\ &= t\frac{t^2}{2} - \frac{t^3}{3} = \frac{t^3}{6} \end{aligned}$$

3.

$$\begin{aligned} 3 * \sin t = \sin t * 3 &= \int_0^t (\sin x)(3)\,dx \\ &= -3\cos x|_{x=0}^{x=t} \\ &= -3(\cos t - \cos 0) \\ &= -3\cos t + 3 \end{aligned}$$

5. From the Convolution table we get

$$\begin{aligned} \sin 2t * e^{3t} &= \frac{1}{3^2 + 2^2}(2e^{3t} - 2\cos 2t - 3\sin 2t) \\ &= \frac{1}{13}(2e^{3t} - 2\cos 2t - 3\sin 2t). \end{aligned}$$

7. From the Convolution table we get

$$\begin{aligned} t^2 * e^{-6t} &= \frac{2}{(-6)^3}(e^{-6t} - (-6 - 6t + (36t^2)/2)) \\ &= \frac{1}{108}(18t^2 - 6t - 6 + e^{-6t}). \end{aligned}$$

9. From the Convolution table we get

$$\begin{aligned} e^{2t} * e^{-4t} = \frac{e^{2t} - e^{-4t}}{2 - (-4)} \\ \frac{1}{6}(e^{2t} - e^{-4t}). \end{aligned}$$

11.

$$\begin{aligned}
\mathcal{L}\left\{e^{at} * \sin bt\right\}(s) &= \frac{1}{s-a}\frac{b}{s^2+b^2} \\
&= \frac{1}{s^2+b^2}\left(\frac{b}{s-a} - \frac{bs+ba}{s^2+b^2}\right) \\
&= \frac{1}{s^2+b^2}\left(b\mathcal{L}\left\{e^{at}\right\} - (b\mathcal{L}\left\{\cos bt\right\} + a\mathcal{L}\left\{\sin bt\right\})\right)
\end{aligned}$$

Thus

$$e^{at} * \sin bt = \frac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt).$$

13. First assume $a \neq b$. Then

$$\begin{aligned}
\mathcal{L}\left\{\sin at * \sin bt\right\} &= \frac{a}{s^2+a^2}\frac{b}{s^2+b^2} \\
&= \frac{1}{b^2-a^2}\left(\frac{ab}{s^2+a^2} - \frac{ab}{s^2+b^2}\right)
\end{aligned}$$

From this it follows that

$$\sin at * \sin bt = \frac{1}{b^2-a^2}(b\sin at - a\sin bt).$$

Now assume $a = b$. Then

$$\mathcal{L}\left\{\sin at * \sin at\right\} = \frac{a^2}{(s^2+a^2)^2}$$

By Table 2.5 in Section 2.5 we get

$$\mathcal{L}\left\{\sin at * \sin at\right\} = \frac{1}{2a}(\sin at - at\cos at).$$

15. First assume $a \neq b$. Then

$$\begin{aligned}
\mathcal{L}\left\{\cos at * \cos bt\right\} &= \frac{s}{s^2+a^2}\frac{s}{s^2+b^2} \\
&= \frac{1}{b^2-a^2}\left(\frac{-a^2}{s^2+a^2} + \frac{b^2}{s^2+b^2}\right)
\end{aligned}$$

From this it follows that

$$\cos at * \cos bt = \frac{1}{b^2-a^2}(-a\sin at + b\sin bt).$$

Now assume $a = b$. Then

$$\begin{aligned} \mathcal{L}\{\cos at * \cos at\} &= \frac{s^2}{(s^2+a^2)^2} \\ &= \frac{1}{s^2+a^2} - \frac{a^2}{(s^2+a^2)^2}. \end{aligned}$$

By Table 2.5 we get

$$\mathcal{L}\{\cos at * \cos at\} = \frac{1}{a}\sin at - \frac{1}{2a}(\sin at - at\cos at) = \frac{1}{2a}(\sin at + at\cos at).$$

17. $f(t) = t^2 * \sin 2t$ so $F(s) = \dfrac{2}{s^3}\dfrac{2}{s^2+4} = \dfrac{4}{s^3(s^2+4)}$.

19. $f(t) = t^3 * e^{-3t}$ so $F(s) = \dfrac{6}{s^4}\dfrac{1}{s+3} = \dfrac{6}{s^4(s+3)}$

21. $f(t) = \sin 2t * \sin 2t$ so $F(s)\dfrac{2}{s^2+2^2}\dfrac{2}{s^2+2^2} = \dfrac{4}{(s^2+4)^2}$

23.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-5)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} \\ &= e^t * e^{5t} \\ &= \frac{e^t - e^{5t}}{1-5} \\ &= \frac{1}{4}(-e^t + e^{5t}) \end{aligned}$$

25.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \\ &= \sin t * \cos t \\ &= \frac{1}{2}t\sin t \end{aligned}$$

27.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{(s-3)(s^2+4)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} * \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= e^{3t} * \sin 2t \\ &= \frac{1}{13}(2e^{3t} - 2\cos 2t - 3\sin 2t)\end{aligned}$$

29.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-b}\right\} \\ &= e^{at} * e^{bt} \\ &= \frac{e^{at} - e^{bt}}{a-b}.\end{aligned}$$

31.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+2}\right\} &= \mathcal{L}^{-1}\{G(s)\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+\sqrt{2}^2}\right\} \\ &= g(t) * \cos(\sqrt{2})t \\ &= \int_0^t g(x)\cos\sqrt{2}(t-x)\,dx\end{aligned}$$

33. We apply the input integral principle twice:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= \int_0^t \sin x\,dx \\ &= -\cos x|_0^t \\ &= -\cos t + 1\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} &= \int_0^t 1 - \cos x\,dx \\ &= t - \sin t\,dx\end{aligned}$$

35. We apply the input integral principle three times:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s+3)}\right\} &= \int_0^t e^{-3x}\,dx \\ &= \left.\frac{e^{-3x}}{-3}\right|_0^t \\ &= \frac{1}{3}(1 - e^{-3t}).\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+3)}\right\} &= \frac{1}{3}\int_0^t 1-e^{-3x}\,dx \\ &= \frac{1}{3}\left(t-\frac{1-e^{-3t}}{3}\right) \\ &= \frac{1}{9}(3t-1+e^{-3t}).\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^3(s+3)}\right\} &= \frac{1}{9}\int_0^t 3x-1+e^{-3x}\,dx \\ &= \frac{1}{9}\left(3\frac{t^2}{2}-t-\frac{e^{-3t}-1}{3}\right) \\ &= \frac{1}{54}(2-6t+9t^2-2e^{-3t})\end{aligned}$$

37. First, $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} = \frac{1}{54}(\sin 3t - 3t\cos 3t)$. Thus

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)^2}\right\} &= \frac{1}{54}\int_0^t (\sin 3x - 3x\cos 3x)\,dx \\ &= \frac{1}{54}\left(-\frac{\cos 3x}{3}-\left(x\sin 3x+\frac{\cos 3x}{3}\right)\right)\Bigg|_0^t \\ &= \frac{1}{54}\left(-\frac{2\cos 3t}{3}-t\sin 3t+\frac{2}{3}\right) \\ &= \frac{1}{162}(-2\cos 3t-3t\sin 3t+2).\end{aligned}$$

Section 3.1

1. no, not linear.

3. no, third order.

5. no, not constant coefficient.

7. yes; $(\boldsymbol{D}^2 - 7\boldsymbol{D} + 10)(y) = 0$, $q(s) = s^2 - 7s + 10$, homogeneous

9. yes; $\boldsymbol{D}^2(y) = -2 + \cos t$, $q(s) = s^2$, nonhomogeneous

11. (a) $\boldsymbol{L}e^t = e^t + 3e^t + 2e^t = 6e^t$
(b) $\boldsymbol{L}e^{-t} = e^{-t} + 3(-e^{-t}) + 2e^{-t} = 0$
(c) $\boldsymbol{L}\sin t = -\sin t + 3(\cos t) + 2\sin t = \sin t - 3\cos t$

13. (a) $\boldsymbol{L}(-4\sin t) = 4\sin t + -4\sin t = 0$
(b) $\boldsymbol{L}(3\cos t) = 3(-\cos t) + 3\cos t = 0$
(c) $\boldsymbol{L}1 = 0 + 1 = 1$

15. e^t and e^{4t} are homogeneous solution so $y_h = c_1e^t + c_2e^{4t}$ are homogeneous solutions for all scalars c_1 and c_2. A particular solution is $y_p = \cos 2t$. Thus $y(t) = y_p(t) + y_h(t) = \cos 2t + c_1e^t + c_2e^{4t}$ where c_1, c_2 are arbitrary constants.

17. From Exercise 15 we have $y(t) = \cos 2t + c_1e^t + c_2e^{4t}$. Since $y' = -2\sin 2t + c_1e^t + 4c_2e^{4t}$ we have

$$\begin{aligned} 1 = y(0) &= 1 + c_1 + c_2 \\ -3 = y'(0) &= c_1 + 4c_2, \end{aligned}$$

from which follows that $c_1 = 1$ and $c_2 = -1$. Thus $y(t) = \cos 2t + e^t - e^{4t}$.

19. $\boldsymbol{L}(e^{rt}) = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = ar^2e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt}$

21. Let $t = a$ be the point ϕ_1 and ϕ_2 are tangent. Then $\phi_1(a) = \phi_2(a)$ and $\phi_1'(a) = \phi_2'(a)$. By the existence and uniqueness theorem $\phi_1 = \phi_2$.

Section 3.2

1. Suppose $c_1t + c_2t^2 = 0$. Evaluating at $t = 1$ and $t = 2$ gives $c_1 + c_2 = 0$ and $2c_1 + 4c_2 = 0$. The simultaneous solution is $c_1 = c_2 = 0$. It follows that $\left\{t, t^2\right\}$ is linearly independent.

3. Since $e^{t+2} = e^te^2$ is a multiple of e^t it follows that $\left\{e^t, e^{t+2}\right\}$ is linearly dependent.

5. Since $\ln t^2 = 2\ln t$ and $\ln t^5 = 5\ln t$ they are multiples of each other and hence linearly dependent

7. Suppose $c_1t + c_2(1/t) = 0$ Evaluating at $t = 1$ and $t = 2$ gives $c_1 + c_2 = 0$ and $2c_1 + c_2/2 = 0$. The simultaneous solution is $c_1 = c_2 = 0$. It follows that $\{t, 1/t\}$ is linearly independent.

9. Suppose $c_1 + c_2(1/t) + c_3(1/t^2) = 0$. Evaluating at $t = 1$, $t = 1/2$, and $t = 1/3$ gives the same system as in the solution to Exercise 8 and hence c_1, c_2 and c_3 are zero. It follows that $\left\{1, 1/t, 1/t^2\right\}$ on $I = (0, \infty)$ is linearly independent.

11. Let $q(s) = s(s-1)(s+1)$. Then $\mathcal{B}_q = \{1, e^t, e^{-t}\}$ which is linearly independent.

13. Let $q(s) = (s-1)^5$. Then $\mathcal{B}_q = \{e^t, te^t, t^2e^t, t^3e^t\}$. Linear independence follows since $\{t^2e^t, t^3e^t, t^4e^t\} \subset \mathcal{B}_q$.

15.

$$w(t, t\ln t) = \det \begin{bmatrix} t & t\ln t \\ 1 & \ln t + 1 \end{bmatrix} = t\ln t + t - t\ln t = t$$

17.

$$w(t^{10}, t^{20}) = \det \begin{bmatrix} t^{10} & t^{20} \\ 10t^9 & 20t^{19} \end{bmatrix} = 20t^{29} - 10t^{29} = 10t^{29}$$

19.

$$\begin{aligned} & w(e^{r_1t}, e^{r_2t}, e^{r_3t}) \\ = \; & \det \begin{bmatrix} e^{r_1t} & e^{r_2t} & e^{r_3t} \\ r_1e^{r_1t} & r_2e^{r_2t} & r_3e^{r_3t} \\ r_1^2e^{r_1t} & r_2^2e^{r_2t} & r_3^2e^{r_3t} \end{bmatrix} \\ = \; & e^{(r_1+r_2+r_3)t} \det \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{bmatrix} \\ = \; & e^{(r_1+r_2+r_3)t}((r_2r_3^2 - r_3r_2^2) - (r_1r_3^2 - r_3r_1^2) + (r_1r_2^2 - r_2r_1^2)) \\ = \; & e^{(r_1+r_2+r_3)t}(r_3 - r_1)(r_3 - r_2)(r_2 - r_1). \end{aligned}$$

The last line requires a little algebra.

21.

$$w(1, t, t^2, t^3) = \det \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 2 & 6t \\ 0 & 0 & 0 & 6 \end{bmatrix} = 12$$

23. Let $q(s) = (s-2)^2$. Then $\mathcal{B}_q = \{e^{2t}, te^{2t}\}$ is linearly independent. We can equation coefficients to get

$$\begin{aligned} 25c_1 \; + \; 10c_2 &= 0 \\ 25c_2 &= 25 \end{aligned}$$

We thus get $c_2 = 1$ and then $c_1 = -10/25 = -2/5$.

25. If $q(s) = s^3$ then $\mathcal{B}_q = \{1, t, t^2\}$ is linearly independent. Thus we can equate coefficients to get

$$\begin{aligned} a_1 &= a_2 \\ 3 &= a_1 \\ -a_2 &= -3 \end{aligned}$$

It follows that $a_1 = 3$ and $a_2 = 3$ is the solution.

27. Observe that $y_1'(t) = 3t^2$ and $y_2'(t) = \begin{cases} -3t^2 & \text{if } t < 0 \\ 3t^2 & \text{if } t \geq 0. \end{cases}$ If $t < 0$ then $w(y_1, y_2)(t) = \begin{pmatrix} t^3 & -t^3 \\ 3t^2 & -3t^2 \end{pmatrix} = 0$. If $t \geq 0$ then $w(y_1, y_2)(t) = \begin{pmatrix} t^3 & t^3 \\ 3t^2 & 3t^2 \end{pmatrix} = 0$. It follows that the Wronskian is zero for all $t \in (-\infty, \infty)$.

Section 3.3

1. The characteristic polynomial is $q(s) = s^2 - s - 2 = (s-2)(s+1)$ so $\mathcal{B}_q = \left\{e^{2t}, e^{-t}\right\}$ and the general solution takes the form $y(t) = c_1 e^{2t} + c_2 e^{-t}$, $c_1, c_2 \in \mathbb{R}$

3. The characteristic polynomial is $q(s) = s^2 + 10s + 24 = (s+6)(s+4)$ so $\mathcal{B}_q = \left\{e^{-6t}, e^{-4t}\right\}$ and the general solution takes the form $y(t) = c_1 e^{-6t} + c_2 e^{-4t}$, $c_1, c_2 \in \mathbb{R}$

5. The characteristic polynomial is $q(s) = s^2 + 8s + 16 = (s+4)^2$ so $\mathcal{B}_q = \left\{e^{-4t}, te^{-4t}\right\}$ and the general solution takes the form $y(t) = c_1 e^{-4t} + c_2 te^{-4t}$, $c_1, c_2 \in \mathbb{R}$

7. The characteristic polynomial is $q(s) = s^2 + 2s + 5 = (s+1)^2 + 4$ so $\mathcal{B}_q = \{e^{-t}\cos 2t, e^{-t}\sin 2t\}$ and the general solution takes the form $y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$, $c_1, c_2 \in \mathbb{R}$

9. The characteristic polynomial is $q(s) = s^2 + 13s + 36 = (s+9)(s+4)$ so $\mathcal{B}_q = \left\{e^{-9t}, e^{-4t}\right\}$ and the general solution takes the form $y(t) = c_1 e^{-9t} + c_2 e^{-4t}$, $c_1, c_2 \in \mathbb{R}$

11. The characteristic polynomial is $q(s) = s^2 + 10s + 25 = (s+5)^2$ so $\mathcal{B}_q = \left\{e^{-5t}, te^{-5t}\right\}$ and the general solution takes the form $y(t) = c_1 e^{-5t} + c_2 te^{-5t}$, $c_1, c_2 \in \mathbb{R}$

13. The characteristic polynomial is $q(s) = s^2 - 1 = (s-1)(s+1)$ so $\mathcal{B}_q = \left\{e^{t}, e^{-t}\right\}$ and the general solution takes the form $y(t) = c_1 e^{2t} + c_2 e^{-t}$. The initial conditions imply that $c_1 + c_2 = 0$ and $c_1 - c_2 = 1$. Solving gives $c_1 = 1/2$ and $c_2 = -1/2$. Thus $y = \frac{e^t - e^{-t}}{2}$

15. The characteristic polynomial is $q(s) = s^2 - 10s + 25 = (s-5)^2$ so $\mathcal{B}_q = \left\{e^{5t}, te^{5t}\right\}$ and the general solution takes the form $y(t) = c_1 e^{5t} + c_2 te^{5t}$. The initial conditions imply that $c_1 = 0$ and $5c_1 + c_2 = 1$. Solving gives $c_1 = 0$ and $c_2 = 1$. Thus $y = te^{5t}$

17. Let $q(s) = (s-3)(s+7) = s^2+4s-21$. Then $\mathcal{B}_q = \left\{e^{3t},\, e^{-7t}\right\}$. $w(e^{3t},\, e^{-7t}) = \det\begin{bmatrix} e^{3t} & e^{-7t} \\ 3e^{3t} & -7e^{-7t}\end{bmatrix} = -10e^{-4t}$. So $K = -10$.

19. Let $q(s) = (s-3)^2 = s^2-6s+9$. Then $\mathcal{B}_q = \left\{e^{3t},\, te^{3t}\right\}$. $w(e^{3t},\, te^{3t}) = \det\begin{bmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (1+3t)e^{3t}\end{bmatrix} = (1+3t)e^{6t} - 3te^{6t} = e^{6t}$. So $K = 1$.

21. Let $q(s) = (s-1)^2+2^2 = s^2-2s+5$. Then $\mathcal{B}_q = \{e^t\cos 2t,\, e^t\sin 2t\}$.

$$\begin{aligned} w(e^t\cos 2t, e^t\sin 2t) &= \det\begin{bmatrix} e^t\cos 2t & e^t\sin 2t \\ e^t(\cos 2t - 2\sin 2t) & e^t(\sin 2t + 2\cos 2t)\end{bmatrix} \\ &= e^{2t}(\sin 2t\cos 2t + 2\cos^2 2t) \\ &\quad -e^{2t}(\cos 2t\sin 2t - 2\sin^2 2t) \\ &= 2e^{2t}. \end{aligned}$$

So $K = 2$.

Section 3.4

1. $q(s)v(s) = (s+1)(s-2)(s-3)$ so $\mathcal{B}_{qv} = \left\{e^{-t},\, e^{2t},\, e^{3t}\right\}$ while $\mathcal{B}_q = \left\{e^{-t},\, e^{2t}\right\}$. Since e^{3t} is the only function in the first set but not in the second $y_p(t) = a_1e^{3t}$.

3. $q(s)v(s) = (s-2)^2(s-3)$ so $\mathcal{B}_{qv} = \left\{e^{2t},\, te^{2t},\, e^{3t}\right\}$ while $\mathcal{B}_q = \left\{e^{2t},\, e^{3t}\right\}$. Since te^{2t} is the only function in the first set but not in the second $y_p(t) = a_1te^{2t}$.

5. $q(s)v(s) = (s-5)^2(s^2+25)$ so $\mathcal{B}_{qv} = \left\{e^{5t},\, te^{5t},\, \cos 5t,\, \sin 5t\right\}$ while $\mathcal{B}_q = \left\{e^{5t},\, te^{5t}\right\}$. Since $\cos 5t$ and $\sin 5t$ are the only functions in the first set that are not in the second $y_p(t) = a_1\cos 5t + a_2\sin 5t$.

7. $q(s)v(s) = (s^2+4)^2$ so $\mathcal{B}_{qv} = \{\cos 2t,\, \sin 2t,\, t\cos 2t,\, t\sin 2t\}$ while $\mathcal{B}_q = \{\cos 2t,\, \sin 2t\}$. Since $t\cos 2t$ and $t\sin 2t$ are the only functions in the first set that are not in the second $y_p(t) = a_1t\cos 2t + a_2t\sin 2t$.

9. $q(s)v(s) = (s^2+4s+5)(s-1)^2$ so $\mathcal{B}_{qv} = \left\{e^t,\, te^t,\, e^{-2t}\cos t,\, e^{-2t}\sin t\right\}$ while $\mathcal{B}_q = \{e^t,\, te^t\}$. Since $e^{-2t}\cos t$ and $e^{-2t}\sin t$ are the only functions in the first set that are not in the second $y_p(t) = a_1e^{-2t}\cos t + a_2e^{-2t}\sin t$.

11. The characteristic polynomial is $q(s) = s^2-3s-10 = (s-5)(s+2)$. Since $\mathcal{L}\left\{7e^{-2t}\right\} = 7/(s+2)$, we set $v(s) = s+2$. Then $q(s)v(s) = (s-5)(s+2)^2$. Since $\mathcal{B}_{qv} = \left\{e^{5t}, e^{-2t}, te^{-2t}\right\}$ and $\mathcal{B}_q = \left\{e^{5t}, e^{-2t}\right\}$ we have $y_p = a_1te^{-2t}$, a test function. Substituting y_p into the differential equation gives $-a_1e^{-2t} = 7e^{-t}$. It follows that $a_1 = -1$. The general solution is $y = -te^{-2t} + c_1e^{-2t} + c_2e^{5t}$.

13. The characteristic polynomial is $q(s) = s^2 + 2s + 1 = (s+1)^2$. Since $\mathcal{L}\{e^{-t}\} = 1/(s+1)$, we set $v(s) = s+1$. Then $q(s)v(s) = (s+1)^3$. Since $\mathcal{B}_{qv} = \{e^{-t}, te^{-t}, t^2e^{-t}\}$ and $\mathcal{B}_q = \{e^{-t}, te^{-t}\}$ we have $y_p = a_1t^2e^{-t}$, a test function. Substituting y_p into the differential equation gives $2a_1e^{-t} = e^{-t}$. It follows that $a_1 = 1/2$. The general solution is $y = \frac{1}{2}t^2e^{-t} + c_1e^{-t} + c_2te^{-t}$.

15. The characteristic polynomial is $q(s) = s^2+4s+5 = (s+2)^2+1$, an irreducible quadratic. Since $\mathcal{L}\{e^{-3t}\} = 1/(s+3)$, we set $v(s) = s+3$. Then $q(s)v(s) = ((s+2)^2+1)(s+3)$. Since $\mathcal{B}_{qv} = \{e^{-2t}\cos t, e^{-2t}\sin t, e^{-3t}\}$ and $\mathcal{B}_q = \{e^{-2t}\cos t, e^{-2t}\sin t\}$ we have $y_p = a_1e^{-3t}$, a test function. Substituting y_p into the differential equation gives $2a_1e^{-3t} = e^{-3t}$. It follows that $a_1 = 1/2$. The general solution is $y = \frac{1}{2}e^{-3t} + c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$.

17. The characteristic polynomial is $q(s) = s^2 - 1 = (s-1)(s+1)$. Since $\mathcal{L}\{t^2\} = 2/s^3$, we set $v(s) = s^3$. Then $q(s)v(s) = (s-1)(s+1)s^3$. Since $\mathcal{B}_{qv} = \{e^t, e^{-t}, 1, t, t^2\}$ and $\mathcal{B}_q = \{e^t, e^{-t}\}$ we have $y_p = a_1 + a_2t + a_3t^2$, a test function. Substituting y_p into the differential equation gives $2a_3 - a_1 - a_2t - a_3t^2 = t^2$. Using linear independence we equate the coefficients to get

$$\begin{aligned} 2a_3 - a_1 &= 0 \\ -a_2 &= 0 \\ -a_3 &= 1 \end{aligned}$$

It follows that $a_3 = -1$, $a_2 = 0$, and $a_1 = -2$. The general solution is $y = -t^2 - 2 + c_1e^t + c_2e^{-t}$.

19. The characteristic polynomial is $q(s) = s^2 - 4s + 4 = (s-2)^2$. Since $\mathcal{L}\{e^{2t}\} = 1/(s-2)$, we set $v(s) = s-2$. Then $q(s)v(s) = (s-2)^3$. Since $\mathcal{B}_{qv} = \{e^{2t}, te^{2t}, t^2e^{2t}\}$ and $\mathcal{B}_q = \{e^{2t}, te^{2t}\}$ we have $y_p = a_1t^2e^{2t}$, a test function. Substituting y_p into the differential equation gives $2a_1e^{2t} = e^{2t}$. It follows that $a_1 = 1/2$. The general solution is $y = \frac{1}{2}t^2e^{2t} + c_1e^{2t} + c_2te^{2t}$

21. The characteristic polynomial is $q(s) = s^2 + 6s + 9 = (s+3)^2$. Since $\mathcal{L}\{25te^{2t}\} = 25/(s-2)^2$, we set $v(s) = (s-2)^2$. Then $q(s)v(s) = (s+3)^2(s-2)^2$. Since $\mathcal{B}_{qv} = \{e^{-3t}, te^{-3t}, e^{2t}, te^{2t}\}$ and $\mathcal{B}_q = \{e^{-3t}, te^{-3t}\}$ we have $y_p = a_1e^{2t} + a_2te^{2t}$, a test function. Substituting y_p into the differential equation gives $(25a_1 + 10a_2)e^{2t} + 25a_2te^{2t} = 25te^{2t}$. Linear independence implies $25a_1 + 10a_2 = 0$ and $25a_2 = 25$. We get $a_2 = 1$ and $a_1 = -2/5$ The general solution is $y = te^{2t} - \frac{2}{5}e^{2t} + c_1e^{-3t} + c_2te^{-3t}$

23. The characteristic polynomial is $q(s) = s^2 + 6s + 13 = (s+3)^2 + 4$, an irreducible quadratic. Since $\mathcal{L}\{e^{-3t}\cos 2t\} = (s+3)/((s+3)^2 +$

4), we set $v(s) = (s+3)^2+4$. Then $q(s)v(s) = ((s+3)^2+4)^2$. Since $\mathcal{B}_{qv} = \left\{e^{-3t}\cos 2t, e^{-3t}\sin 2t, te^{-3t}\cos 2t, te^{-3t}\sin 2t\right\}$ and $\mathcal{B}_q = \left\{e^{-3t}\cos 2t, e^{-3t}\sin 2t\right\}$ we have $y_p = a_1 te^{-3t}\cos 2t + a_2 te^{-3t}\sin 2t$, a test function. Substituting y_p into the differential equation gives (after a long calculation) $-4a_1e^{-3t}\sin 2t + 4a_2e^{-3t}\cos 2t = e^{-3t}\cos 2t$. It follows that $-4a_1 = 0$ and $4a_2 = 1$. Thus $a_1 = 0$ and $a_2 = 1/4$. The general solution is $y = \frac{1}{4}te^{-3t}\sin(2t) + c_1e^{-3t}\cos(2t) + c_2e^{-3t}\sin(2t)$.

25. The characteristic polynomial is $q(s) = s^2 - 5s - 6 = (s-6)(s+1)$. Since $\mathcal{L}\left\{e^{3t}\right\} = 1/(s-3)$, we set $v(s) = s-3$. Then $q(s)v(s) = (s-6)(s+1)(s-3)$. Since $\mathcal{B}_{qv} = \left\{e^{6t}, e^{-t}, e^{3t}\right\}$ and $\mathcal{B}_q = \left\{e^{6t}, e^{-t}\right\}$ we have $y_p = a_1e^{3t}$, a test function. Substituting y_p into the differential equation gives $-12a_1e^{3t} = e^{3t}$. It follows that $a_1 = -1/12$. The general solution is $y = \frac{-1}{12}e^{3t} + c_1e^{6t} + c_2e^{-t}$. Since $y' = \frac{-1}{4}e^{3t} + 6c_1e^{6t} - c_2e^{-t}$ the initial condition imply

$$\begin{array}{rcrcrl} \frac{-1}{12} & + & c_1 & + & c_2 & = 2 \\ \frac{-1}{4} & + & 6c_1 & - & c_2 & = 1 \end{array}$$

It is easy to calculate that $c_1 = 10/21$ and $c_2 = 135/84$. Thus $y = \frac{-1}{12}e^{3t} + \frac{10}{21}e^{6t} + \frac{135}{84}e^{-t}$.

27. The characteristic polynomial is $q(s) = s^2+1$. Since $\mathcal{L}\left\{10e^{2t}\right\} = 10/(s-2)$, we set $v(s) = s-2$. Then $q(s)v(s) = (s^2+1)(s-2)$. Since $\mathcal{B}_{qv} = \left\{\cos t, \sin t, e^{2t}\right\}$ and $\mathcal{B}_q = \{\cos t, \sin t\}$ we have $y_p = a_1e^{2t}$, a test function. Substituting y_p into the differential equation gives $5a_1e^{2t} = 10e^{2t}$ and hence $a_1 = 2$. The general solution is $y = 2e^{2t} + c_1\cos t + c_2\sin t$. Since $y' = 4e^{2t} - c_1\sin t + c_2\cos t$ the initial conditions imply

$$\begin{array}{rcrrcl} 2 & + & c_1 & & = & 0 \\ 4 & + & & c_2 & = & 0 \end{array}$$

and so $c_1 = -2$ and $c_2 = -4$. Thus $y = 2e^{2t} - 2\cos t - 4\sin t$.

Section 3.5

1. The characteristic polynomial is $q(s) = s^2 - 4 = (s-2)(s+2)$ and $\mathcal{L}\left\{e^{-6t}\right\} = 1/(s+6)$. Thus

$$\mathcal{L}\{y\} = \frac{1}{(s-2)(s+2)(s+6)} = \frac{\frac{1}{32}}{s+6} + \frac{p(s)}{(s-2)(s+2)}.$$

A particular solution is $y_p = \frac{1}{32}e^{-6t}$ and the general solution is $y = \frac{1}{32}e^{-6t} + c_1e^{2t} + c_2e^{-2t}$

3. The characteristic polynomial is $q(s) = s^2 + 5s + 6 = (s+2)(s+3)$ and $\mathcal{L}\left\{e^{-2t}\right\} = 1/(s+2)$. Thus

$$\mathcal{L}\{y\} = \frac{1}{(s+2)^2(s+3)} = \frac{1}{(s+2)^2} + \frac{p(s)}{(s+2)(s+3)}.$$

A particular solution is $y_p = te^{-2t}$ and the general solution is $y = te^{-2t} + c_1e^{-2t} + c_2e^{-3t}$

5. The characteristic polynomial is $q(s) = s^2 + 2s - 8 = (s-2)(s+4)$ and $\mathcal{L}\left\{6e^{-4t}\right\} = 6/(s+4)$. Thus

$$\mathcal{L}\{y\} = \frac{6}{(s-2)(s+4)^2} = \frac{-1}{(s+4)^2} + \frac{p(s)}{(s-2)(s+4)}.$$

A particular solution is $y_p = -te^{-4t}$ and the general solution is $y = -te^{-4t} + c_1e^{2t} + c_2e^{-4t}$

7. The characteristic polynomial is $q(s) = s^2 + 6s + 9 = (s+3)^2$ and $\mathcal{L}\left\{25e^{2t}\right\} = 25/((s-2)^2)$. Thus

$$\mathcal{L}\{y\} = \frac{25}{(s-2)^2(s+3)^2} = \frac{1}{(s-2)^2} - \frac{2}{5}\frac{1}{s-2} + \frac{p(s)}{(s+3)^2}.$$

A particular solution is $y_p = te^{2t} - \frac{2}{5}e^{2t}$ and the general solution is $y = te^{2t} - \frac{2}{5}e^{2t} + c_1e^{-3t} + c_2te^{-3t}$

9. The characteristic polynomial is $q(s) = s^2 - 8s + 25 = (s-4)^2 + 9$ and $\mathcal{L}\left\{36te^{4t}\sin 3t\right\} = 216(s-4)/((s-4)^2+9)^2$. Thus

$$\mathcal{L}\{y\} = \frac{216(s-4)}{((s-4)^2+9)^3}.$$

This is a partial fraction. Table 2.5 gives $y = -3t^2e^{4t}\cos 3t + te^{4t}\sin 3t$. A particular solution is $y_p = -3t^2e^{4t}\cos 3t + te^{4t}\sin 3t$ and the general solution is $y = -3t^2e^{4t}\cos 3t + te^{4t}\sin 3t + c_1e^{4t}\cos 3t + c_2e^{4t}\sin 3t$

11. The characteristic polynomial is $q(s) = s^2 + 2s + 1 = (s+1)^2$ and $\mathcal{L}\{\cos t\} = s/(s^2+1)$. Thus

$$\mathcal{L}\{y\} = \frac{s}{(s+1)^2(s^2+1)} = \frac{1}{2}\frac{1}{s^2+1} + \frac{p(s)}{(s+1)^2}.$$

A particular solution is $y_p = \frac{1}{2}\sin t$ and the general solution is $y = \frac{1}{2}\sin t + c_1e^{-t} + c_2te^{-t}$

Section 3.6

1. The force is 16 lbs. A length of 6 inches is 1/2 ft. The spring constant is $k = 16/(1/2) = 32$ lbs/ft.

3. The force exerted by the mass is $40 \cdot 9.8 = 392$ N. Thus $k = 392/.8 = 490$ N/m.

5. The force is 4 lbs and the velocity is 1/2 ft per second. So $\mu =$ Force/velocity $= \frac{4}{1/2} = 8$ lbs s/ft.

7. Let x be the force. Then $100 = x/4$ so $x = 400$ lbs.

9. The mass is $m = 6$. The spring constant is given by $k = 2/.1 = 20$. The damping constant is $\mu = 0$. Since no external force is mentioned we may assume it is zero. The initial conditions are $y(0) = .1$ m and $y'(0) = 0$. The following equation

$$6y'' + 20y = 0, \qquad y(0) = .1, \quad y'(0) = 0$$

represents the model for the motion of the body. The characteristic polynomial is $q(s) = 6s^2 + 20 = 6(s^2 + \sqrt{10/3}^2)$. Thus $y = c_1 \cos \sqrt{10/3}\, t + c_2 \sin \sqrt{10/3}\, t$. The initial conditions imply $c_1 = 1/10$ and $c_2 = 0$. Thus

$$y = \frac{1}{10} \cos \sqrt{10/3}\, t.$$

The motion is undamped free or simple harmonic motion. Since y is written in the form $y = A \cos \omega t + \phi$ we can read off the amplitude, frequency, and phase shift; they are $A = 1/10$, $\beta = \sqrt{10/3}$, and $\phi = 0$.

11. The mass is $m = 16/32 = 1/2$ slugs. The spring constant k is given by $k = 16/(6/12) = 32$. The damping constant is given by $\mu = 4/2 = 2$. Since no external force is mentioned we may assume it is zero. The initial conditions are $y(0) = 1$ and $y'(0) = 1$. The following equation $\frac{1}{2}y'' + 2y' + 32y = 0,\ y(0) = 1,\ y'(0) = 1$ models the motion of the body. The characteristic polynomial is $q(s) = \frac{1}{2}s^2 + 2s + 32 = \frac{1}{2}(s^2 + 4s + 64) = \frac{1}{2}((s+2)^2 + \sqrt{60}^2)$. Thus

$$y = c_1 e^{-2t} \cos \sqrt{60}t + c_2 e^{-2t} \sin \sqrt{60}t.$$

The initial conditions imply $c_1 = 1$ and $c_2 = 3/\sqrt{60}$. Thus

$$y = e^{-2t} \cos \sqrt{60}t + \frac{3}{\sqrt{60}} e^{-2t} \sin \sqrt{60}t.$$

The discriminant of the characteristic equation is $D = 2^2 - 4\cdot(1/2)\cdot 32 = -60 < 0$ so the motion is underdamped free motion. Let $A = \sqrt{1+\left(\frac{3}{\sqrt{60}}\right)^2} = \sqrt{23/20}$. If $\tan\phi = 3/\sqrt{60} = \sqrt{60}/20$ then $\phi \approx .3695$. We can write

$$y = \sqrt{\frac{23}{20}} e^{-2t}\cos(\sqrt{60}t+\phi).$$

13. The mass is $m = 2/32 = 1/16$ slug. The spring constant k is given by $k = 2/(4/12) = 6$ and the damping constant is $\mu = 0$. The initial conditions are $y(0) = 0$ and $y'(0) = 8/12 = 2/3$. The equation $\frac{1}{16}y'' + 6y = 0$ or equivalently $y'' + 96y = 0$ with initial conditions $y(0) = 0,\ y'(0) = 2/3$ models the motion of the body. The characteristic polynomial is $q(s) = s^2 + 96$ so $y = c_1\cos\sqrt{96}t + c_2\sin\sqrt{96}t$. The initial conditions imply $c_1 = 0$ and $c_2 = \frac{\sqrt{6}}{36}$. Thus $y = \frac{\sqrt{6}}{36}\sin\sqrt{96}t = \frac{\sqrt{6}}{36}\cos\left(\sqrt{96}t - \frac{\pi}{2}\right)$. The motion is undamped free or simple harmonic motion so the mass crosses equilibrium.

15. By the quadratic formula the roots of $q(s) = ms^2 + \mu s + k$ are

$$s = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m} = \frac{-\mu}{2m} \pm \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}}.$$

If the discriminant $D = \mu^2 - 4mk$ is negative then the roots are complex and the real part is $\frac{-\mu}{2m}$ which is negative. If the discriminant is zero then $\frac{-\mu}{2m}$ is a negative double root. If the discriminant is positive then both roots are real and distinct. It is enough to show that the larger of the two, $r = \frac{-\mu}{2m} + \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}}$ is negative. Let $p = \frac{\mu}{2m} + \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}}$ and observe that it is positive. Further,

$$\begin{aligned} rp &= \left(\frac{-\mu}{2m} + \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}}\right)\left(\frac{\mu}{2m} + \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}}\right) \\ &= -\frac{\mu^2}{4m^2} + \frac{\mu^2}{4m^2} - \frac{k}{m} \\ &= -\frac{k}{m} < 0 \end{aligned}$$

Since $p > 0$ it follows the $r < 0$.

It follows that a solution to $my'' + \mu y' + ky = 0$ is of the following form

1. $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ where r_1 and r_2 are negative.
2. $y = (c_1 + c_2 t)e^{rt}$ where r is negative
3. $y = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$ where α is negative

In each case $\lim_{t\to\infty} y(t) = 0$.

SECTION 4.1

1. yes; $(\boldsymbol{D}^3 - 3\boldsymbol{D})y = e^t$, order 3, $q(s) = s^3 - 3s$, nonhomogeneous

3. no, because of the presence of y^4

5. (a) $\boldsymbol{L}(e^{2t}) = 8e^{2t} - 4(2e^{2t}) = 0$
(b) $\boldsymbol{L}(3^{-2t} = -8e^{-2t} - 4(-2e^{-2t}) = 0$
(c) $\boldsymbol{L}(2) = 0 - 4(0) = 0$

7. (a) $\boldsymbol{L}e^{-t} = e^{-t} + 5e^{-t} + 4e^{-t} = 10e^{-t}$
(b) $\boldsymbol{L}\cos t = \cos t + 5(-\cos t) + 4\cos t = 0$
(c) $\boldsymbol{L}\sin 2t = 16\sin 2t + 5(-4\sin 2t) + 4\sin 2t = 0$

9. e^{2t}, e^{-2t}, and 1 are homogeneous solution so $y_h = c_1e^{2t} + c_2e^{-2t} + c_3$ are homogeneous solutions for all scalars c_1, c_2, and c_3. A particular solution is $y_p = te^{2t}$. Thus $y(t) = y_p(t) + y_h(t) = te^{2t} + c_1e^{2t} + c_2e^{-2t} + c_3$ where c_1, c_2, and c_3 are arbitrary constants

11. From Exercise 9 we have $y(t) = y_p(t) + y_h(t) = te^{2t} + c_1e^{2t} + c_2e^{-2t} + c_3$. Since

$$\begin{aligned} y &= te^{2t} + c_1e^{2t} + c_2e^{-2t} + c_3 \\ y' &= (1+2t)e^{2t} + 2c_1e^{2t} - 2c_2e^{-2t} \\ y'' &= (4+4t)e^{2t} + 4c_1e^{2t} + 4c_2e^{-2t} \end{aligned}$$

we have

$$\begin{aligned} 2 = y(0) &= c_1 + c_2 + c_3 \\ -1 = y'(0) &= 1 + 2c_1 - 2c_2 \\ 16 = y''(0) &= 4 + 4c_1 + 4c_2, \end{aligned}$$

from which follows that $c_1 = 1$, $c_2 = 2$, and $c_3 = -1$. Thus $y(t) = te^{2t} + e^{2t} + 2e^{-2t} - 1$

Section 4.2

1. The characteristic polynomial is $q(s) = s^3 - 1 = (s-1)(s^2+s+1) = (s-1)((s+1/2)^2+3/4)$. Thus $\mathcal{B}_q = \left\{ e^t, e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t, e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t \right\}$. It follows that $y(t) = c_1 e^{-t} + c_2 e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + c_3 e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

3. The characteristic polynomial is $q(s) = s^4 - 1 = (s^2-1)(s^2+1) = (s-1)(s+1)(s^2+1)$. Thus $\mathcal{B}_q = \{e^t, e^{-t}, \cos t, \sin t\}$. It follows that $y(t) = c_1 e^t + c_2 e^{-t} + c_3 \sin t + c_4 \cos t$

5. The characteristic polynomial is $q(s) = s^4 - 5s^2 + 4 = (s^2-1)(s^2-4) = (s-1)(s+1)(s-2)(s+2)$. Thus $\mathcal{B}_q = \left\{ e^t, e^{-t}, e^{2t}, e^{-2t} \right\}$. It follows that $y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t}$

7. The characteristic polynomial is $q(s) = (s+2)(s^2+25)$. Thus $\mathcal{B}_q = \left\{ e^{-2t}, \cos 5t, \sin 5t \right\}$. It follows that $y(t) = c_1 e^{-2t} + c_2 \cos 5t + c_3 \sin 5t$.

9. The characteristic polynomial is $q(s) = (s+3)(s-1)(s+3)^2 = (s-1)(s+3)^3$. Thus $\mathcal{B}_q = \left\{ e^t, e^{-3t}, te^{-3t}, t^2 e^{-3t} \right\}$. It follows that $y(t) = c_1 e^t + c_2 e^{-3t} + c_3 t e^{-3t} + c_4 t^2 e^{-3t}$.

11. The characteristic polynomial is $q(s) = s^4 - 1 = (s^2-1)(s^2+1) = (s-1)(s+1)(s^2+1)$. Thus $\mathcal{B}_q = \{e^t, e^{-t}, \cos t, \sin t\}$. It follows that $y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$. Since

$$\begin{aligned} y(t) &= c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t \\ y'(t) &= c_1 e^t - c_2 e^{-t} - c_3 \sin t + c_4 \cos t \\ y''(t) &= c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t \\ y'''(t) &= c_1 e^t - c_2 e^{-t} + c_3 \sin t - c_4 \cos t \end{aligned}$$

we have

$$\begin{aligned} -1 = y(0) &= c_1 + c_2 + c_3 \\ 6 = y'(0) &= c_1 - c_2 + c_4 \\ -3 = y''(0) &= c_1 + c_2 - c_3 \\ 2 = y'''(0) &= c_1 - c_2 - c_4 \end{aligned}$$

from which we get $c_1 = 1$, $c_2 = -3$, $c_3 = 1$, and $c_4 = 2$. Hence $y = e^t - 3e^{-t} + \cos t + 2\sin t$

Section 4.3

1. Since $q(s) = s^3 - s = s(s-1)(s+1)$ we have $\mathcal{B}_q = \{1, e^t, e^{-t}\}$ and since $q(s)v(s) = s(s-1)(s+1)^2$ we have $\mathcal{B}_{qv} = \{1, e^t, e^{-t}, te^{-t}\}$. Thus $\mathcal{B}_{qv} \setminus \mathcal{B}_q = \{te^{-t}\}$ and $y = cte^{-t}$ is the test function.

3. $q(s) = s(s-1)(s+1)$ we have $\mathcal{B}_q = \{1, e^t, e^{-t}\}$ and since $q(s)v(s) = s(s-1)(s+1)(s-2)$ we have $\mathcal{B}_{qv} = \left\{1, e^t, e^{-t}, e^{2t}\right\}$. Thus $\mathcal{B}_{qv} \setminus \mathcal{B}_q = \left\{e^{2t}\right\}$ and $y = ce^{2t}$ is the test function.

5. We have $q(s) = s^3 - s = s(s-1)(s+1)$ and $\mathcal{L}\{e^t\} = \frac{1}{s-1}$. Let $v(s) = s-1$. Then $q(s)v(s) = s(s-1)^2(s+1)$, $\mathcal{B}_q = \{1, e^t, e^{-t}\}$ and $\mathcal{B}_{qv} = \{1, e^t, te^t, e^{-t}\}$ and $\mathcal{B}_{qv} \setminus \mathcal{B}_q = \{te^t\}$. It follows that $y = cte^t$ is the test function. Since

$$\begin{aligned} y &= cte^t \\ y' &= c(1+t)e^t \\ y'' &= c(2+t)e^t \\ y''' &= c(3+t)e^t \end{aligned}$$

we have $c(3+t)e^t - c(1+t)e^t = e^t$. Simplifying we get $2ce^t = e^t$ which implies $c = 1/2$. It follows that $y = \frac{1}{2}te^t + c_1e^{-t} + c_2e^t + c_3$

7. We have $q(s) = s^4 - 5s^2 + 4 = (s^2-1)(s^2-4) = (s-1)(s+1)(s-2)(s+2))$ and $\mathcal{L}\left\{e^{2t}\right\} = \frac{1}{s-2}$. Let $v(s) = s-2$. Then $q(s)v(s) = (s-1)(s+1)(s-2)^2(s+2)$, $\mathcal{B}_q = \left\{e^t, e^{-t}, e^{2t}, e^{-2t}\right\}$ and $\mathcal{B}_{qv} = \left\{e^t, e^{-t}, e^{2t}, te^{2t}, e^{-2t}\right\}$. Thus $\mathcal{B}_{qv} \setminus \mathcal{B}_q = \left\{te^{2t}\right\}$. It follows that $y = cte^{2t}$ is the test function and

$$\begin{aligned} y &= cte^{2t} \\ y' &= c(1+2t)e^{2t} \\ y'' &= c(4+4t)e^{2t} \\ y''' &= c(12+8t)e^{2t} \\ y^{(4)} &= c(32+16t)e^{2t}. \end{aligned}$$

Substituting into the differential equation and simplifying gives $12ce^{2t} = e^{2t}$. We thus get $c = 1/12$. It follows that $y = \frac{1}{12}te^{2t} + c_1e^t + c_2e^{-t} + c_3e^{2t} + c_4e^{-2t}$

9. We have $q(s) = s^3 - s = s(s-1)(s+1)$ and $\mathcal{L}\{e^t\} = \frac{1}{s-1}$. Thus $Y(s) = \frac{1}{s(s+1)(s-1)^2}$. One iteration of the partial fraction decomposition algorithm gives

Incomplete $(s-1)$**-chain**	
$\frac{1}{s(s+1)(s-1)^2}$	$\frac{1/2}{(s-1)^2}$
$\frac{p(s)}{s(s+1)(s-1)}$	

It follows that $y_p = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = \frac{1}{2}te^t$ and the general solution is $y = \frac{1}{2}te^t + c_1e^{-t} + c_2e^t + c_3$

11. We have $q(s) = s(s^2+4)$ and $\mathcal{L}\{t\} = \frac{1}{s^2}$. Thus $Y(s) = \frac{1}{s^3(s^2+4)}$. The partial fraction decomposition algorithm gives

Incomplete s**-chain**	
$\frac{1}{s^3(s^2+4)}$	$\frac{1/4}{s^3}$
$\frac{-s/4}{s^2(s^2+4)}$	0
$\frac{-1/4}{s(s^2+4)}$	

It follows that $y_p = \frac{t^2}{8}$ and the general solution is $y = \frac{t^2}{8} + c_1 + c_2\cos 2t + c_3\sin 2t$.

13. We have $q(s) = s^3 - s^2 + s - 1 = (s-1)(s^2+1)$ and $\mathcal{L}\{4\cos t\} = \frac{4s}{s^2+1}$. Thus $Y(s) = \frac{4s}{(s-1)(s^2+1)^2}$. The partial fraction decomposition algorithm gives

Incomplete s^2+1**-chain**	
$\frac{4s}{(s-1)(s^2+1)^2}$	$\frac{-2s+2}{(s^2+1)^2}$
$\frac{p(s)}{(s-1)(s^2+1)}$	

It follows from Table 2.9 that $y_p = (-t\sin t + \sin t - t\cos t)$. But since $\sin t$ is a homogeneous solution we can write the general solution as $y = -t(\sin t + \cos t) + c_1 e^t + c_2 \cos t + c_2 \sin t$.

Section 4.4

1. Here $\boldsymbol{L}_1 = \boldsymbol{D} - 6$ and $\boldsymbol{L}_2 = \boldsymbol{D}$. It is easy to see that $\boldsymbol{L} = q(\boldsymbol{D})$, where $q(s) = s^2 - 6s + 8 = (s-2)(s-4)$. Therefore y_1 and y_2 are linear combinations of $\mathcal{B}_q = \left\{e^{2t}, e^{4t}\right\}$. Next we recursively extend the initial values to derivatives of order 1 to get

$$\begin{aligned} y_1(0) &= 2 & y_2(0) &= -1 \\ y_1'(0) &= 16 & y_2'(0) &= 4 \end{aligned}$$

If $y = c_1 e^{2t} + c_2 e^{4t}$ then

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & y(0) \\ 2c_1 & + & 4c_2 & = & y'(0) \end{array}$$

For y_1 we get

$$\begin{array}{rcrcr} c_1 & + & c_2 & = & 2 \\ 2c_1 & + & 4c_2 & = & 16 \end{array}$$

which gives $c_1 = -4$ and $c_2 = 6$. Thus $y_1(t) = -4e^{2t} + 6e^{4t}$. For y_2 we get

$$\begin{array}{rcrcr} c_1 & + & c_2 & = & -1 \\ 2c_1 & + & 4c_2 & = & 4 \end{array}$$

which gives $c_1 = -4$ and $c_2 = 3$. Thus $y_2(t) = -4e^{2t} + 3e^{4t}$.

3. Here $\boldsymbol{L}_1 = \boldsymbol{D}$ and $\boldsymbol{L}_2 = \boldsymbol{D}$. It is easy to see that $\boldsymbol{L} = q(\boldsymbol{D})$, where $q(s) = s^2+4$. Therefore y_1 and y_2 are linear combinations of $\mathcal{B}_q = \{\cos 2t, \sin 2t\}$. Next we recursively extend the initial values to derivatives of order 1 to get

$$\begin{aligned} y_1(0) &= 1 & y_2(0) &= -1 \\ y_1'(0) &= -2 & y_2'(0) &= 2 \end{aligned}$$

If $y = c_1 \cos 2t + c_2 \sin 2t$ then

$$\begin{array}{rrcl} c_1 & & = & y(0) \\ & 2c_2 & = & y'(0) \end{array}$$

For y_1 we get

$$\begin{array}{rrcr} c_1 & & = & 1 \\ & 2c_2 & = & -2 \end{array}$$

which gives $c_1 = 1$ and $c_2 = -1$. Thus $y_1(t) = \cos 2t - \sin 2t$. For y_2 we get

$$\begin{aligned} c_1 \qquad &= -1 \\ 2c_2 &= \ \ 2 \end{aligned}$$

which gives $c_1 = -1$ and $c_2 = 1$. Thus and $y_2(t) = -\cos 2t + \sin 2t$.

5. Here $\boldsymbol{L}_1 = \boldsymbol{D}+4$ and $\boldsymbol{L}_2 = \boldsymbol{D}^2 - 6\boldsymbol{D}+23$. It is easy to see that $\boldsymbol{L} = q(\boldsymbol{D})$, where $q(s) = (s+4)(s^2-6s+23)-90 = (s^2-1)(s-2) = (s+1)(s-1)(s-2)$. Therefore y_1 and y_2 are linear combinations of $\mathcal{B}_q = \left\{e^{-t}, e^t, e^{2t}\right\}$. Next we recursively extend the initial values to derivatives of order 2 to get

$$\begin{aligned} y_1(0) &= 0 & y_2(0) &= 2 \\ y_1'(0) &= 20 & y_2'(0) &= 2 \\ y_1''(0) &= -60 & y_2''(0) &= -34. \end{aligned}$$

If $y = c_1 e^{-t} + c_2 e^t + c_3 e^{2t}$ then

$$\begin{array}{rcrcrcl} c_1 & + & c_2 & + & c_3 & = & y(0) \\ -c_1 & + & c_2 & + & 2c_3 & = & y'(0) \\ c_1 & + & c_2 & + & 4c_3 & = & y''(0). \end{array}$$

For y_1 we get

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & + & c_3 & = & 0 \\ -c_1 & + & c_2 & + & 2c_3 & = & 20 \\ c_1 & + & c_2 & + & 4c_3 & = & -60. \end{array}$$

which gives $c_1 = -20$, $c_2 = 40$, and $c_3 = -20$. Thus $y_1(t) = -20e^{-t} + 40e^t - 20e^{2t}$. For y_2 we get

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & + & c_3 & = & 2 \\ -c_1 & + & c_2 & + & 2c_3 & = & 2 \\ c_1 & + & c_2 & + & 4c_3 & = & -34. \end{array}$$

which gives $c_1 = -6$, $c_2 = 20$, and $c_3 = -12$. Thus and $y_2(t) = -6e^{-t} + 20e^t - 12e^{2t}$.

7. Here $\boldsymbol{L}_1 = \boldsymbol{D}^2 + 2\boldsymbol{D} + 6$ and $\boldsymbol{L}_2 = \boldsymbol{D}^2 - 2\boldsymbol{D} + 6$. It is easy to see that $\boldsymbol{L} = q(\boldsymbol{D})$, where $q(s) = (s^2 + 2s + 6)(s^2 - 2s + 6) - 45 = s^4 + 8s^2 - 9 = (s^2 - 1)(s^2 + 9) = (s - 1)(s + 1)(s^2 + 9)$. Therefore y_1 and y_2 are linear combinations of $\mathcal{B}_q = \{e^t, e^{-t}, \cos 3t, \sin 3t\}$. Next we recursively extend the initial values to derivatives of order 3 to get

$$\begin{aligned} y_1(0) &= 0 & y_2(0) &= 6 \\ y_1'(0) &= 0 & y_2'(0) &= 6 \\ y_1''(0) &= 30 & y_2''(0) &= -24 \\ y_1'''(0) &= -30 & y_2'''(0) &= -84 \end{aligned}$$

If $y = c_1 e^t + c_2 e^{-t} + c_3 \cos 3t + c_4 \sin 3t$ then

$$\begin{array}{rcrcrcrcl} c_1 & + & c_2 & + & c_3 & & & = & y(0) \\ c_1 & - & c_2 & & & + & 3c_4 & = & y'(0) \\ c_1 & + & c_2 & - & 9c_3 & & & = & y''(0) \\ c_1 & - & c_2 & & & - & 27c_4 & = & y'''(0) \end{array}$$

For y_1 we get

$$\begin{array}{rcrcrcrcr} c_1 & + & c_2 & + & c_3 & & & = & 0 \\ c_1 & - & c_2 & & & + & 3c_4 & = & 0 \\ c_1 & + & c_2 & - & 9c_3 & & & = & 30 \\ c_1 & - & c_2 & & & - & 27c_4 & = & -30 \end{array}$$

which gives $c_1 = 0$, $c_2 = 3$, $c_3 = -3$, and $c_4 = 1$. Thus $y_1(t) = 3e^{-t} - 3\cos 3t + \sin 3t$. For y_2 we get

$$\begin{array}{rcrcrcrcr} c_1 & + & c_2 & + & c_3 & & & = & 6 \\ c_1 & - & c_2 & & & + & 3c_4 & = & 6 \\ c_1 & + & c_2 & - & 9c_3 & & & = & -24 \\ c_1 & - & c_2 & & & - & 27c_4 & = & -84 \end{array}$$

which gives $c_1 = 0$, $c_2 = 3$, $c_3 = 3$ and $c_4 = 3$. Thus and $y_2(t) = 3e^{-t} + 3\cos 3t + 3\sin 3t$.

9. Here $a = 2$, $b = 1$, and $c = 2$ and the coupled system that describes the motion is given by

$$\begin{aligned} y_1'' + 3y_1 &= y_2 \\ y_2'' + 2y_2 &= 2y_1. \end{aligned}$$

Let $\boldsymbol{L}_1 = \boldsymbol{D}^2 + 3$ and $\boldsymbol{L}_2 = \boldsymbol{D}^2 + 2$. Then y_1 and y_2 a solutions to $q(\boldsymbol{D})y = 0$, where $q(s) = (s^2+3)(s^2+2)-2 = s^4+5s^2+4 = (s^2+1)(s^2+4)$. Thus y_1 and y_2 a linear combinations of $\mathcal{B}_q = \{\cos t, \sin t, \cos 2t, \sin 2t\}$. Next we recursively extend the initial values to derivatives of order 3 to get

$$\begin{array}{rcrcrcr} y_1(0) & = & 3 & \quad & y_2(0) & = & 0 \\ y_1'(0) & = & 3 & & y_2'(0) & = & 0 \\ y_1''(0) & = & -9 & & y_2''(0) & = & 6 \\ y_1'''(0) & = & -9 & & y_2'''(0) & = & 6 \end{array}$$

If $y = c_1 \cos t + c_2 \sin t + c_3 \cos 2t + c_4 \sin 2t$ then

$$\begin{array}{rcrcrcrcl} c_1 & & & + & c_3 & & & = & y(0) \\ & & c_2 & & & + & 2c_4 & = & y'(0) \\ -c_1 & & & - & 4c_3 & & & = & y''(0) \\ & & -c_2 & & & - & 8c_4 & = & y'''(0) \end{array}$$

For y_1 we get

$$\begin{array}{rcrcrcrcr} c_1 & & & + & c_3 & & & = & 3 \\ & & c_2 & & & + & 2c_4 & = & 3 \\ -c_1 & & & - & 4c_3 & & & = & -9 \\ & & -c_2 & & & - & 8c_4 & = & -9 \end{array}$$

which gives $c_1 = 1$, $c_2 = 1$, $c_3 = 2$, and $c_4 = 1$. Thus $y_1(t) = \cos t + \sin t + 2\cos 2t + \sin 2t$. For y_2 we get

$$\begin{array}{rcrcrcrcr} c_1 & & & + & c_3 & & & = & 0 \\ & & c_2 & & & + & 2c_4 & = & 0 \\ -c_1 & & & - & 4c_3 & & & = & 6 \\ & & -c_2 & & & - & 8c_4 & = & 6 \end{array}$$

which gives $c_1 = 2$, $c_2 = 2$, $c_3 = -2$ and $c_4 = -1$. Thus $y_2(t) = 2\cos t + 2\sin t - 2\cos 2t - \sin 2t$.

11. 1. We begin by taking the Laplace transform of each equation above to get

$$\begin{aligned} q_1(s)Y_1(s) - p_1(s) &= \lambda_1 Y_2(s) \\ q_2(s)Y_2(s) - p_2(s) &= \lambda_2 Y_1(s) \end{aligned}$$

which can be rewritten:

$$\begin{aligned} q_1(s)Y_1(s) - \lambda_1 Y_2(s) &= p_1(s) \\ q_2(s)Y_2(s) - \lambda_2 Y_1(s) &= p_2(s). \end{aligned}$$

In matrix form this becomes

$$\begin{pmatrix} q_1(s) & -\lambda_1 \\ -\lambda_2 & q_2(s) \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix}$$

2. The inverse of the coefficient matrix is

$$\begin{pmatrix} q_1(s) & -\lambda_1 \\ -\lambda_2 & q_2(s) \end{pmatrix}^{-1} = \frac{1}{q_1(s)q_2(s) - \lambda_1\lambda_2} \begin{pmatrix} q_2(s) & \lambda_1 \\ \lambda_2 & q_1(s) \end{pmatrix}$$

and therefore

$$\begin{aligned} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} &= \frac{1}{q_1(s)q_2(s) - \lambda_1\lambda_2} \begin{pmatrix} q_2(s) & \lambda_1 \\ \lambda_2 & q_1(s) \end{pmatrix} \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix} \\ &= \frac{1}{q_1(s)q_2(s) - \lambda_1\lambda_2} \begin{pmatrix} p_1(s)q_2(s) + \lambda_1 p_2(s) \\ p_2(s)q_1(s) + \lambda_2 p_1(s) \end{pmatrix}. \end{aligned}$$

13. We first take the Laplace transform of each equation to get

$$\begin{aligned} sY_1(s) - 2 - Y_1 &= -2Y_2(s) \\ sY_2(s) - (-2) - Y_2(s) &= 2Y_1(s). \end{aligned}$$

We associate the Y_1 and Y_2. In matrix form we get

$$\begin{pmatrix} s-1 & 2 \\ -2 & s-1 \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

By matrix inversion we get

$$\begin{aligned} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} &= \begin{pmatrix} s-1 & 2 \\ -2 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \\ &= \frac{1}{(s-1)^2+4} \begin{pmatrix} s-1 & -2 \\ 2 & s-1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \\ &= \frac{1}{(s-1)^2+4} \begin{pmatrix} 2s+2 \\ -2s+6 \end{pmatrix} = \begin{pmatrix} \frac{2(s-1)+4}{(s-1)^2+2^2} \\ \frac{-2(s-1)+4}{(s-1)^2+2^2} \end{pmatrix} \end{aligned}$$

We now get by Laplace inversion

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 2e^t \cos 2t + 2e^t \sin 2t \\ -2e^t \cos 2t + 2e^t \sin 2t \end{pmatrix}.$$

15. We first take the Laplace transform of each equation to get

$$\begin{aligned} sY_1(s) - 1 + 2Y_1 &= 5Y_2(s) \\ s^2Y_2(s) - 3 - 2(sY_2(s)) + 5Y_2(s) &= 2Y_1(s). \end{aligned}$$

We associate the Y_1 and Y_2. In matrix form we get

$$\begin{pmatrix} s+2 & -5 \\ -2 & s^2-2s+5 \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

By matrix inversion we get

$$\begin{aligned} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} &= \begin{pmatrix} s+2 & -5 \\ -2 & s^2-2s+5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ &= \frac{1}{s^3+s} \begin{pmatrix} s^2-2s+5 & 5 \\ 2 & s+2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ &= \frac{1}{s(s^2+1)} \begin{pmatrix} s^2-2s+20 \\ 3s+8 \end{pmatrix} = \begin{pmatrix} \frac{20}{s} - 19\frac{s}{s^2+1} - 2\frac{1}{s^2+1} \\ \frac{8}{s} - 8\frac{s}{s^2+1} + 3\frac{1}{s^2+1} \end{pmatrix} \end{aligned}$$

We now get by Laplace inversion

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 20 - 19\cos t - 2\sin t \\ 8 - 8\cos t + 3\sin t \end{pmatrix}.$$

Section 4.5

1. The only characteristic mode is e^{-5t}. Thus the zero-input response is $y(t) = ce^{-5t}$. The initial condition $a = y(0) = 10$ implies $c = 10$. Thus $y(t) = 10e^{-5t}$, The characteristic value is -5, to the left of the imaginary axis. Hence the system is stable.

3. The characteristic polynomial is $q(s) = s^2 - 4s + 3 = (s-3)(s-1)$. The characteristic modes is $\{e^t, e^{3t}\}$. Thus $y(t) = c_1e^t + c_2e^{3t}$. The initial condition $\boldsymbol{a} = (2, 4) = (y(0), y'(0))$ implies $c_1 = 1$ and $c_2 = 1$. Thus the zero-input response is $y(t) = e^t + e^{3t}$. The characteristic value are $1, 3$ and both lie to the right of the imaginary axis. Hence the system is unstable.

5. The characteristic polynomial is $q(s) = s^2 + 4s + 5 = (s+2)^2 + 1$. The characteristic modes are $\{e^{-2t}\cos t, e^{-2t}\sin t\}$. Thus $y(t) = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$. The initial condition $\boldsymbol{a} = (0, 1) = (y(0), y'(0))$ implies $c_1 = 0$ and $c_2 = 1$. Thus the zero-input response is $y(t) = e^{-2t}\sin t$. The characteristic value are $-2+i, -2-i$ and both lie to the left of the imaginary axis. Hence the system is stable.

7. The characteristic polynomial is $q(s) = s^2 + 6s + 9 = (s+3)^2$. The characteristic modes are $\{e^{-3t}, te^{-3t}\}$. Thus $y(t) = c_1e^{-3t} + c_2te^{-3t}$. The initial condition $\boldsymbol{a} = (1, 1) = (y(0), y'(0))$ implies $c_1 = 1$ and $c_2 = 4$. Thus the zero-input response is $y(t) = e^{-3t} + 4te^{-3t}$. The characteristic value is -3 with multiplicity 2 lies to the left of the imaginary axis. Hence the system is stable.

9. The characteristic polynomial is $q(s) = s^2 - 2s + 2 = (s-1)^2 + 1$. The characteristic modes are $\{e^t\cos t, e^t\sin t\}$. Thus $y(t) = c_1e^t\cos t + c_2e^t\sin t$. The initial condition $\boldsymbol{a} = (1, 2) = (y(0), y'(0))$ implies $c_1 = 1$ and $c_2 = 1$. Thus the zero-input response is $y(t) = e^t\cos t + e^t\sin t$. The characteristic values are $\{1+i, 1-i\}$ and lie to the right of the imaginary axis. Hence the system is unstable.

11. The characteristic polynomial is $q(s) = (s+1)(s^2+1)$. The characteristic modes are $\{e^{-t}, \cos t, \sin t\}$. Thus $y(t) = c_1e^{-t} + c_2\cos t + c_3\sin t$. The initial condition $\boldsymbol{a} = (1, -1, 1) = (y(0), y'(0), y''(0))$ implies $c_1 = 1$, $c_2 = 0$, and $c_3 = 0$. Thus the zero-input response is $y(t) = e^{-t}$. The characteristic value are -1, i, and $-i$. The system is then marginally stable.

13. The characteristic mode is e^{-t} so $y(t) = ce^{-t}$. For the unit impulse we have $y(0) = 1$ and this implies $c = 1$. Thus $y(t) = e^{-t}$

15. The characteristic polynomial is $q(s) = s^2 - 4 = (s-2)(s+2)$ and hence the characteristic modes are $\left\{e^{2t}, e^{-2t}\right\}$. Hence, $y(t) = c_1e^{2t} + c_2e^{-2t}$. For the unit impulse we have $y(0) = 0$ and $y'(0) = 1$ and this implies $c_1 = 1/4$ and $c_2 = -1/4$. Thus $y(t) = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$.

17. The characteristic polynomial is $q(s) = s^3 + s = s(s^2+1)$. The characteristic modes are $\{1, \cos 2t, \sin 2t\}$. For the unit impulse we have $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 1$ and this implies $c_1 = 1$, $c_2 = -1$ and $c_3 = 0$. Thus $y(t) = 1 - \cos(t)$.

19. Since f is bounded there is an M such that $|f(t)| \leq M$ for all $t \geq 0$. We then have

$$\begin{aligned} \left|t^k e^{\alpha t} \cos \beta t * f(t)\right| &= \left|\int_0^t x^k e^{\lambda x} \cos \beta x f(t-x)\, dx\right| \\ &\leq \int_0^t x^k e^{\alpha x} \left|f(t-x)\right|\, dx \\ &\leq M \int_0^t x^k e^{\alpha x}\, dx \\ &= M(C + p(t)e^{\alpha t}), \end{aligned}$$

where C and $p(t)$ are as in Exercise 18, which also implies that $t^k e^{\alpha t} \cos \beta t * f$ is bounded. The argument for $t^k e^{\alpha t} \sin \beta t * f$ is the same.

Section 5.1

1. No, it is not linear because of the presence of the product $y'y$.

3. yes, nonhomogeneous, yes

5. yes, nonhomogeneous, no

7. yes, nonhomogeneous, no

9. No, it is not linear because of the presence of $\sin y$.

11. yes, homogeneous, no

13.
1. $\boldsymbol{L}(\frac{1}{t}) = t^2(2t^{-3}) + t(-t^{-2}) - t^{-1} = (2 - 1 - 1)t^{-1} = 0$
2. $\boldsymbol{L}(1) = t^2(0) + t(0) - 1 = -1$
3. $\boldsymbol{L}(t) = t^2(0) + t(1) - t = 0$
4. $\boldsymbol{L}(t^r) = t^2 r(r-1)t^{r-2} + t(rt^{r-1}) - t^r = (r^2 - 1)t^r$

15. $y_p' = C(2t - t^2)e^{-t}$ and $y_p'' = C(t^2 - 4t + 2)e^{-t}$. Thus

$$\begin{aligned} t^2 y_p'' + t y_p' - y_p &= C(t^3 - 4t^2 + 2t)e^{-t} + C(-t^3 + 3t^2 - 2t)e^{-t} + C(-t^2)e^{-t} \\ &= C(-2t^2)e^{-t} \end{aligned}$$

The equation $C(-2t^2)e^{-t} = t^2 e^{-t}$ implies $C = \frac{-1}{2}$.

17. If $y = e^{-t}$ then $y' = -e^{-t}$ and $y'' = e^{-t}$ so that $\boldsymbol{L}y = (t-1)e^{-t} - t(-e^{-t}) + e^{-t} = 2te^{-t}$. Parts (1) follows. If $y = e^t$ then $\boldsymbol{L}y = (t-1)(e^t) - t(e^t) + (e^t) = 0$. It follows that $y = e^t$ is a solution to $\boldsymbol{L}y = 0$. If $y = t$ then $y' = 1$ and $y'' = 0$. Thus $\boldsymbol{L}y = (t-1)(0) - t(1) + t = 0$. Part (2) now follows. By linearity every function of the form $y(t) = e^{-t} + c_1 e^t + c_2 t$ is a solution to $\boldsymbol{L}y = 2te^{-t}$, where c_1 and c_2 are constants. If we want a solution to $\boldsymbol{L}(y) = 2te^{-t}$ with $y(0) = a$ and $y'(0) = b$, then we need to solve for c_1 and c_2: Since $y(t) = e^{-t} + c_1 e^t + c_2 t$ we have $y'(t) = -e^{-t} + c_1 e^t + c_2$. Hence,

$$\begin{aligned} a &= y(0) = 1 + c_1 \\ b &= y'(0) = -1 + c_1 + c_2. \end{aligned}$$

These equations give $c_1 = a - 1$ and $c_2 = b - a + 2$. Particular choices of a and b give the answers for Part (3).

(3)a. $y(t) = e^{-t} - e^t + 2t$

(3)b. $y(t) = e^{-t} + (0)e^t + (1)t = e^{-t} + t$

(3)c. $y(t) = e^{-t} + -e^t + 3t$

(3)d. $y(t) = e^{-t} + (a-1)e^t + (b - a + 2)t$

19. Write the equation in the standard form:

$$y'' + \frac{3}{t}y' - \frac{1}{t^2}y = t^2.$$

The forcing function is continuous on $\mathbb{R}$ while the coefficient functions, $\frac{3}{t}$ and $-\frac{1}{t^2}$, are continuous except at $t = 0$. Thus the largest intervals of common continuity are $(0, \infty)$ and $(-\infty, 0)$. Since the initial conditions are given at $t_0 = -1$ it follows from Theorem 6 that the interval $(-\infty, 0)$ is the largest interval with a unique solution.

21. Write the equation in the standard form:

$$y'' + \frac{y}{\sin t} = \frac{\cos t}{\sin t}.$$

The intervals of continuity are of the form $(k\pi, (k+1)\pi)$, $k \in \mathbb{Z}$. Since $t_0 = \frac{\pi}{2}$ it follows that the maximal interval for a unique solution is $(0, \pi)$.

23. The common interval of continuity of the coefficient functions is $(3, \infty)$ and $t_0 = 10$ is in this interval.

25. The initial condition occurs at $t = 0$ which is precisely where $a_2(t) = t^2$ has a zero. Theorem 6 does not apply.

27. The assumptions say that $y_1(t_0) = y_2(t_0)$ and $y_1'(t_0) = y_2'(t_0)$. Both y_1 and y_2 therefore satisfies the same initial conditions. By the uniqueness part of Theorem 6 $y_1 = y_2$.

Section 5.2

1. dependent; $2t$ and $5t$ are multiples of each other.

3. independent; If $c_1 \ln t + c_2 t \ln t = 0$ then evaluating at $t = e$ and $t = e^2$ gives $c_1 + ec_2 = 0$ and $2c_1 + 2e^2 c_2 = 0$. These equations imply that c_1 and c_2 are both zero so $\{\ln t, t\ln t\}$ is linearly independent.

5. independent, If $c_1 \ln 2t + c_2 \ln 5t = 0$ then evaluating at $t = 1$ and $t = e$ gives $(\ln 2)c_1 + (\ln 5)c_2 = 0$ and $(1 + \ln 2)c_1 + (\ln 5 + 1)c_2 = 0$. These equations imply that c_1 and c_2 are both zero so $\{\ln t, t\ln t\}$ is linearly independent.

7. $f_1'(t) = e^t - 1$ and $f_2''(t) = e^t$. Thus $(t-1)f_1'' - tf_1' + f_1 = (t-1)(e^t) - t(e^t - 1) + e^t - t = 0$. Similarly, $f_2'(t) = 1$ and $f_2''(t) = 0$. Thus $(t-1)f_2'' - tf_2' + f_2 = -t(1) + t = 0$. Now,

$$w(t) = \begin{vmatrix} e^t - t & t \\ e^t - 1 & 1 \end{vmatrix} = e^t - t - (e^t - 1)t = (1-t)e^t.$$

On the other hand the coefficient function of y' in the standard form of the differential equation is $a_1(t) = -\frac{t}{t-1} = -1 - \frac{1}{t-1}$ Integrating gives $\int_0^t -1 - \frac{1}{x-1}\,dx = -x - \ln|x-1|\,|_0^t = -t + \ln(1-t)$, (since $x - 1 < 0$) and $e^{-\int_0^t a_1(x)\,dx} = e^t(1-t)$. At $t = 0$ we have $w(1) = 1$ so Abel's formula is verified. It follows from Proposition 4 that f_1 and f_2 are linearly independent. By Theorem 2 the solution set is $\{c_1(e^t - t) + c_2 t : c_1, c_2 \in \mathbb{R}\}$

9. $f_1'(t) = \frac{-2\sin(2\ln t)}{t}$, $f_1''(t) = \frac{2\sin(2\ln t) - 4\cos(2\ln t)}{t^2}$, $f_2'(t) = \frac{2\cos(2\ln t)}{t}$, and $f_2''(t) = \frac{-4\sin(2\ln t) - 2\cos(2\ln t)}{t^2}$. Thus $t^2 f_1'' + tf_1' + 4f_1 = 2\sin(2\ln t) - 4\cos(2\ln t) - 2\sin(2\ln t) + 4\cos(2\ln t) = 0$. Similarly, $t^2 f_2'' + tf_2' + 4f_2 = -4\sin(2\ln t) - 2\cos(2\ln t) + 2\cos(2\ln t) + 4\sin(2\ln t) = 0$. Now,

$$w(t) = \begin{vmatrix} \cos(2\ln t) & \sin(2\ln t) \\ \dfrac{-2\sin(2\ln t)}{t} & \dfrac{2\cos(2\ln t)}{t} \end{vmatrix} = \frac{4}{t}.$$

On the other hand the coefficient function of y' in the standard form of the differential equation is $a_1(t) = \frac{1}{t}$ Integrating gives $\int_1^t \frac{1}{x}\,dx = \ln t$ and $e^{-\int_0^t a_1(x)\,dx} = 1/t$. At $t = 1$ we have $w(1) = 1$ so Abel's formula is verified. It follows from Proposition 4 that f_1 and f_2 are linearly independent. By Theorem 2 the solution set is $\{c_1 \cos(2\ln t) + c_2 \sin(2\ln t) : c_1, c_2 \in \mathbb{R}\}$

11. 1. Suppose $at^3 + b\left|t^3\right| = 0$ on $(-\infty, \infty)$. Then for $t = 1$ and $t = -1$ we get

$$\begin{aligned} a + b &= 0 \\ -a + b &= 0. \end{aligned}$$

These equations imply $a = b = 0$. So y_1 and y_2 are linearly independent.

2. Observe that $y_1'(t) = 3t^2$ and $y_2'(t) = \begin{cases} -3t^2 & \text{if } t < 0 \\ 3t^2 & \text{if } t \geq 0. \end{cases}$ If $t < 0$ then $w(y_1, y_2)(t) = \begin{pmatrix} t^3 & -t^3 \\ 3t^2 & -3t^2 \end{pmatrix} = 0$. If $t \geq 0$ then $w(y_1, y_2)(t) = \begin{pmatrix} t^3 & t^3 \\ 3t^2 & 3t^2 \end{pmatrix} = 0$. It follows that the Wronskian is zero for all $t \in (-\infty, \infty)$.
3. The condition that the coefficient function $a_2(t)$ be nonzero in Theorem 2 and Proposition 4 is essential. Here the coefficient function, t^2, of y'' is zero at $t = 0$, so Proposition 4 does not apply on $(-\infty, \infty)$. The largest open intervals on which t^2 is nonzero are $(-\infty, 0)$ and $(0, \infty)$. On each of these intervals y_1 and y_2 are linearly dependent.
4. Consider the cases $t < 0$ and $t \geq 0$. The verification is then straightforward.
5. Again the condition that the coefficient function $a_2(t)$ be nonzero is essential. The Uniqueness and Existence theorem does not apply.

Section 5.3

1. The indicial polynomial is $Q(s) = s^2 + s - 2 = (s + 2)(s - 1)$. There are two distinct roots 1 and -2. The fundamental set is $\left\{t, t^{-2}\right\}$. The general solution is $y(t) = c_1 t + c_2 t^{-2}$.

3. The indicial polynomial is $Q(s) = 9s^2 - 6s + 1 = (3s-1)^2$. There is one root, 1/3, with multiplicity 2. The fundamental set is $\left\{t^{\frac{1}{3}}, t^{\frac{1}{3}} \ln t\right\}$. The general solution is $y(t) = c_1 t^{\frac{1}{3}} + c_2 t^{\frac{1}{3}} \ln t$.

5. The indicial polynomial is $Q(s) = 4s^2 - 4s + 1 = (2s-1)^2$. The root is $\frac{1}{2}$ with multiplicity 2. The fundamental set is $\left\{t^{\frac{1}{2}}, t^{\frac{1}{2}} \ln t\right\}$. The general solution is $y(t) = c_1 t^{\frac{1}{2}} + c_2 t^{\frac{1}{2}} \ln t$.

7. The indicial polynomial is $Q(s) = s^2 + 6s + 9 = (s+3)^2$. The root is -3 with multiplicity 2. The fundamental set is $\left\{t^{-3}, t^{-3} \ln t\right\}$. The general solution is $y(t) = c_1 t^{-3} + c_2 t^{-3} \ln t$.

9. The indicial polynomial is $Q(s) = s^2 - 4 = (s-2)(s+2)$. There are two distinct roots, 2 and -2. The fundamental set is $\left\{t^2, t^{-2}\right\}$. The general solution is $y(t) = c_1 t^2 + c_2 t^{-2}$.

11. The indicial polynomial is $Q(s) = s^2 - 4s + 13 = (s-2)^2 + 9$. There are two complex roots, $2 + 3i$ and $2 - 3i$. The fundamental set is $\left\{t^2 \cos(3 \ln t), t^2 \sin(3 \ln t)\right\}$. The general solution is $y(t) = c_1 t^2 \cos(3 \ln t) + c_2 t^2 \sin(3 \ln t)$.

13. The indicial polynomial is $Q(s) = 4s^2 - 4s + 1 = (2s-1)(2s-1)$. There is a double root, $r = \frac{1}{2}$. The fundamental set is $\left\{t^{\frac{1}{2}}, t^{\frac{1}{2}} \ln t\right\}$. The general solution is $y(t) = c_1 t^{\frac{1}{2}} + c_2 t^{\frac{1}{2}} \ln t$. The initial conditions imply

$$\begin{aligned} c_1 &= 2 \\ \frac{1}{2} c_1 + c_2 &= 0. \end{aligned}$$

Thus $c_1 = 2$ and $c_2 = -1$. Hence $y = 2t^{1/2} - t^{1/2} \ln t$

15. The coefficient functions for the given equation in standard form are $a_1(t) = -4/t$ and $a_2(t) = 6/t^2$ both of which are not defined at the initial condition $t_0 = 0$. Thus the uniqueness and existence theorem does not guarantee a solution. In fact, the condition that $y'(0)$ exist presupposes that y is defined near $t = 0$. For t positive the indicial polynomial is $Q(s) = s^2 - 5s + 6 = (s-6)(s+1)$ and therefore $y(t) = c_1 t^6 + c_2 t^{-1}$. The only way that y can be extended to $t = 0$ is that $c_2 = 0$. In this case $y(t) = c_1 t^6$ cannot satisfy the given initial conditions. Thus, no solution is possible.

Section 5.4

1. By L'Hospital's rule $\lim_{t\to 0} \frac{e^{bt}-e^{at}}{t} = b - a$. So Theorem 4 applies and gives

$$\begin{aligned}
\mathcal{L}\left\{\frac{e^{bt}-e^{at}}{t}\right\}(s) &= \int_s^\infty \frac{1}{\sigma - b} - \frac{1}{\sigma - a}\, d\sigma \\
&= \lim_{M\to\infty} \ln\left(\frac{M-b}{M-a}\right) - \ln\left(\frac{s-b}{s-a}\right) \\
&= \ln\left(\frac{s-a}{s-b}\right)
\end{aligned}$$

3. Apply L'Hospital's rule twice to get $\lim\limits_{t\to\infty} 2\dfrac{\cos bt - \cos at}{t^2} = a^2 - b^2$. Now use Exercise 2 to get

$$\begin{aligned}
\mathcal{L}\left\{2\frac{\cos bt - \cos at}{t^2}\right\}(s) &= \int_s^\infty \ln\left(\frac{\sigma^2+a^2}{\sigma^2+b^2}\right) d\sigma \\
&= \lim_{M\to\infty}\left(\int_s^M \ln(\sigma^2+a^2)\, d\sigma - \int_s^M \ln(\sigma^2+b^2)\, d\sigma\right).
\end{aligned}$$

We now use two facts from calculus:

1. $\int \ln(x^2+a^2)\, dx = x\ln(x^2+a^2) - 2x + 2a\tan^{-1}(x/a) + C$
2. $\lim\limits_{x\to\infty} x\ln\left(\frac{x^2+a^2}{x^2+b^2}\right) = 0$

The first fact is shown by integration by parts and the second fact is shown by L'Hospitals rule. We now get (after some simplifications)

$$\mathcal{L}\left\{2\frac{\cos bt - \cos at}{t^2}\right\}(s) = s\ln\left(\frac{s^2+b^2}{s^2+a^2}\right) + 2a\tan^{-1}\left(\frac{a}{s}\right) - 2b\tan^{-1}\left(\frac{b}{s}\right)$$

5. Applying the Laplace transform we get

$$Y' + \frac{3s+2}{s^2+s}Y = \frac{2y_0}{s^2+s}.$$

The integrating factor is $I = s^2(s+1)$; we get $Y(s) = \frac{y_0}{s+1} + \frac{C}{s^2(s+1)}$. Laplace inversion gives

$$\begin{aligned}
y(t) &= y_0 e^{-t} + C(t - 1 + e^{-t}) \\
&= (y_0 + C)e^{-t} + C(t-1).
\end{aligned}$$

Let $c_1 = C$ and $c_2 = y_0 + C$ to get $y(t) = c_1 e^{-t} + c_2(t-1)$.

7. Applying the Laplace transform we get $Y'(s) = \dfrac{-y_0}{(s+2)^2}$ and therefore $Y(s) = \dfrac{y_0}{s+2} + C$. However, since $\lim_{s\to\infty} Y(s) = 0$ we must have $C = 0$. Hence $y(t) = y_0 e^{-2t}$.

9. Applying the Laplace transform we get $Y'(s) + \dfrac{6s}{s^2+1} Y(s) = 0$. An integrating factor is $I = (s^2+1)^3$. We then get $Y(s) = \dfrac{C}{(s^2+1)^3}$, and $y(t) = (C/8)\left((3-t^2)\sin t - 3t\cos t\right)$

11. Apply the Laplace transform to get $Y'(s) = \dfrac{-y_0}{s(s-1)} = y_0\left(\dfrac{1}{s} - \dfrac{1}{s-1}\right)$. Then $Y(s) = y_0 \ln\left(\dfrac{s}{s-1}\right) + C$. Take $C = 0$ since $\lim_{s\to\infty} Y(s) = 0$. Hence $y(t) = y_0 \dfrac{e^t - 1}{t}$.

13. Apply the Laplace transform, simplify, and get $Y'(s) = \dfrac{-y_0}{(s^2-5s+6)} = y_0\left(\dfrac{1}{s-2} - \dfrac{1}{s-3}\right)$. Then $Y(s) = y_0 \ln\left(\dfrac{s-2}{s-3}\right) + C$. Take $C = 0$. Then $y(t) = y_0\left(\dfrac{e^{3t} - e^{2t}}{t}\right)$.

15. Apply the Laplace transform, simplify, and get $Y'(s) = \dfrac{-sy_0}{s(s^2+1)} - \dfrac{2y_1}{s(s^2+1} = -y_0 \dfrac{1}{s^2+1} - 2y_1\left(\dfrac{1}{s} - \dfrac{s}{s^2+1}\right)$. Integrating gives $Y(s) = -y_0 \tan^{-1}(s) + y_1 \ln\left(\dfrac{s^2+1}{s^2}\right) + C$. Since $\lim_{s\to\infty} Y(s) = 0$ we must have $C = y_0\dfrac{\pi}{2}$ and hence $Y(s) = y_0 \tan^{-1}\left(\dfrac{1}{s}\right) + y_1 \ln\left(\dfrac{s^2+1}{s^2}\right)$. Therefore $y(t) = y_0 \dfrac{\sin t}{t} + 2y_1 \dfrac{1-\cos t}{t}$

17. We use the formula

$$\frac{d^n}{dt^n}(f(t)g(t)) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dt^k} f(t) \cdot \frac{d^{n-k}}{dt^{n-k}} g(t).$$

Observe that

$$\frac{d^k}{dt^k} e^{-t} = (-1)^k e^{-t}$$

and

$$\frac{d^{n-k}}{dt^{n-k}} t^n = n(n-1)\cdots(k+1)t^k.$$

It now follows that

$$\begin{aligned}
&\frac{1}{n!}e^t \frac{d^n}{dt^n}(e^{-t}t^n)\\
=\ &\frac{1}{n!}e^t \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dt^k}e^{-t}\frac{d^{n-k}}{dt^{n-k}}t^n\\
=\ &e^t \sum_{k=0}^{n} \binom{n}{k}(-1)^k e^{-t}\frac{n(n-1)\cdots(k+1)}{n!}t^k\\
=\ &\sum_{k=0}^{n}(-1)^k \binom{n}{k}\frac{t^k}{k!}\\
=\ &\ell_n(t).
\end{aligned}$$

21. Hint: Take the Laplace transform of each side. Use the previous exercise and the binomial theorem.

23. We compute the Laplace transform of both sides. We'll do a piece at a time.

$$\begin{aligned}
&\mathcal{L}\left\{(2n+1)\ell_n\right\}(s)\\
=\ &(2n+1)\frac{(s-1)^n}{s^{n+1}}\\
=\ &\frac{(s-1)^{n-1}}{s^{n+2}}(2n+1)(s(s-1)).
\end{aligned}$$

$$\begin{aligned}
&\mathcal{L}\left\{-t\ell_n\right\}(s)\\
=\ &\left(\frac{(s-1)^n}{s^{n+1}}\right)'\\
=\ &\frac{(s-1)^{n-1}}{s^{n+2}}(n+1-s).
\end{aligned}$$

$$\begin{aligned}
&-n\mathcal{L}\left\{\ell_{n-1}\right\}(s)\\
=\ &-n\frac{(s-1)^{n-1}}{s^n}\\
=\ &\frac{(s-1)^{n-1}}{s^{n+2}}(-ns^2).
\end{aligned}$$

We have written each so that the common factor is $\frac{(s-1)^{n-1}}{s^{n+2}}$. The coefficients are

$$\begin{aligned} & n+1-s+(2n+1)(s(s-1))-ns^2 \\ = \; & (n+1)(s^2-2s+1) \\ = \; & (n+1)(s-1)^2 \end{aligned}$$

The right hand side is now

$$\begin{aligned} & \frac{1}{n+1}\left((n+1)(s-1)^2\frac{(s-1)^{n-1}}{s^{n+2}}\right) \\ = \; & \frac{(s-1)^{n+1}}{s^{n+2}} \\ = \; & \mathcal{L}\left\{\ell_{n+1}\right\}(s). \end{aligned}$$

Taking the inverse Laplace transform completes the verification.

25. First of all $\int_0^\infty e^{-t}\ell_n(t)\,dt = \mathcal{L}\left\{\ell_n\right\}(1) = 0$. Thus

$$\begin{aligned} & \int_t^\infty e^{-x}\ell_n(x)\,dt \\ = \; & -\int_0^t e^{-x}\ell_n(x)\,dx \\ = \; & -e^{-t}\int_0^\infty e^{t-x}\ell_n(x)\,dx \\ = \; & -e^{-t}(e^t * \ell_n(t)). \end{aligned}$$

By the convolution theorem

$$\begin{aligned} & \mathcal{L}\left\{e^t * \ell_n\right\}(s) \\ = \; & \frac{1}{s-1}\frac{(s-1)^n}{s^{n+1}} \\ = \; & \frac{(s-1)^{n-1}}{s^{n+1}} \\ = \; & \frac{(s-1)^{n-1}}{s^n}\left(1-\frac{s-1}{s}\right) \\ = \; & \frac{(s-1)^{n-1}}{s^n} - \frac{(s-1)^n}{s^{n+1}} \\ = \; & \mathcal{L}^{-1}\left\{\ell_{n-1}(t)\right\} - \mathcal{L}^{-1}\left\{\ell_n(t)\right\}. \end{aligned}$$

It follows by inversion that $e^t * \ell_n = \ell_{n-1} - \ell_n$ and substituting this formula into the previous calculation gives the needed result.

Section 5.5

1. Let $y_2(t) = t^2 u(t)$. Then $t^4 u'' + t^3 u' = 0$, which gives $u' = t^{-1}$ and $u(t) = \ln t$. Substituting gives $y_2(t) = t^2 \ln t$. The general solution can be written $y(t) = c_1 t^2 + c_2 t^2 \ln t$.

3. Let $y_2(t) = t^{\frac{1}{2}} u(t)$. Then $4t^{\frac{5}{2}} u'' + 4t^{\frac{3}{2}} u' = 0$ leads to $u' = 1/t$ and hence $u(t) = \ln t$. Thus $y_2(t) = \sqrt{t} \ln t$. The general solution can be written $y(t) = c_1 \sqrt{t} + c_2 \sqrt{t} \ln t$.

5. Let $y_2(t) = tu(t)$. Then u satisfies $t^3 u'' - t^3 u' = 0$. Thus $u' = e^t$ and $u = e^t$. It follows that $y_2(t) = te^t$ is a second independent solution. The general solution can be written $y(t) = c_1 t + c_2 te^t$.

7. Let $y_2(t) = u(t) \sin t^2$. Then $u(t)$ satisfies $t \sin t^2 u'' + (4t^2 \cos t^2 - \sin t^2) u' = 0$ and hence $\dfrac{u''}{u'} = \dfrac{1}{t} - 4t \dfrac{\cos t^2}{\sin t^2}$. It follows that $u' = t \csc^2 t^2$ and therefore $u(t) = \frac{-1}{2} \cot t^2$. We now get $y_2(t) = \frac{-1}{2} \cos t^2$. The general solution can be written $y(t) = c_1 \sin t^2 + c_2 \cos t^2$.

9. Let $y_2(t) = u(t) \tan t$. Then $u'' \tan t + 2u' \sec^2 t = 0$ which gives $u' = \cot^2 t = \csc^2 t - 1$. Hence $u = -\cot t - t$ and $y_2(t) = -1 - t \tan t$. The general solution can be written $y(t) = c_1 \tan t + c_2 (1 + t \tan t)$.

11. The functions $\tan t$ and $\sec t$ are continuous except at points of the form $\frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$. We will work in the interval $(-\pi/2, \pi/2)$. Let $y_2(t) = u(t) \tan t$. Then $u'' \tan t + u'(\tan^2 t + 2) = 0$ and hence $\frac{u''}{u'} = -\tan t - 2 \cot t$. It follows that $\ln |u'| = \ln |\cos t| - 2 \ln |\sin t|$ and thus $u' = \cos t \sin^{-2} t$. Further $u(t) = \frac{-1}{\sin t}$ and we have $y_2(t) = -\sec t$. The general solution can be written $y(t) = c_1 \tan t + c_2 \sec t$.

13. Let $y_2 = u \frac{\sin 2t}{1+\cos 2t}$. Then $u(t)$ satisfies $u'' \sin 2t + 4u' = 0$ and hence $\frac{u''}{u'} = -4 \csc 2t$. We now get $\ln u' = 2 \ln |\csc 2t + \cot 2t|$. Thus $u' = (\csc 2t + \cot 2t)^2 = \csc^2 2t + 2 \csc 2t \cot 2t + \cot^2 2t = 2 \csc^2 2t + 2 \csc 2t \cot 2t - 1$. By integrating we get $u = -\cot 2t - \csc 2t - t = -\frac{1+\cos 2t}{\sin 2t} - t$. It now follows that $y_2 = -1 - \frac{t \sin 2t}{1+\cos 2t}$. The general solution can be written $y(t) = c_1 \frac{\sin 2t}{1+\cos 2t} + c_2 \left(1 + \frac{t \sin 2t}{1+\cos 2t}\right)$.

15. Let $y_2(t) = (1-t^2) u(t)$. Substitution gives $(1-t^2)^2 u'' - 4t(1-t^2) u' = 0$ and hence $\frac{u''}{u'} = -2 \frac{-2t}{1-t^2}$. From this we get $u' = \frac{1}{(1-t^2)^2}$. Integrating u' by partial fractions give $u = \frac{1}{2} \frac{t}{1-t^2} + \frac{1}{4} \ln \left(\frac{1+t}{1-t}\right)$ and hence

$$y_2(t) = \frac{1}{2} t + \frac{1}{4} (1-t^2) \ln \left(\frac{1+t}{1-t}\right).$$

The general solution can be written

$$y = c_1(1-t^2) + c_2\left(\frac{1}{2}t + \frac{1}{4}(1-t^2)\ln\left(\frac{1+t}{1-t}\right)\right).$$

Section 5.6

1. $\sin t$ and $\cos t$ form a fundamental set for the homogeneous solutions. Let $y_p(t) = u_1\cos t + u_2\sin t$. Then the matrix equation $\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}u_1'\\ u_2'\end{pmatrix} = \begin{pmatrix}0\\ \sin t\end{pmatrix}$ implies $u_1'(t) = -\sin^2 t = \frac{1}{2}(\cos 2t - 1)$ and $u_2'(t) = \cos t\sin t = \frac{1}{2}(\sin 2t)$. Integration give $u_1(t) = \frac{1}{4}(\sin(2t) - 2t) = \frac{1}{2}(\sin t\cos t - t)$ and $u_2(t) = \frac{-1}{4}\cos 2t = \frac{-1}{4}(2\cos 2t - 1)$. This implies $y_p(t) = \frac{1}{4}\sin t - \frac{1}{2}t\cos t$. Since $\frac{1}{4}\sin t$ is a homogeneous solution we can write the general solution in the form $y(t) = \frac{-1}{2}t\cos t + c_1\cos t + c_2\sin t$. We observe that a particular solution is the imaginary part of a solution to $y'' + y = e^{it}$. We use the incomplete partial fraction method and get $Y(s) = \frac{1}{(s-i)^2(s+i)}$. This can be written $Y(s) = \frac{1}{2i}\frac{1}{(s-i)^2} + \frac{p(s)}{(s-i)(s+i)}$. From this we get $y_p(t) = \operatorname{Im}\left(\frac{1}{2i}\mathcal{L}^{-1}\{\frac{1}{(s-i)^2}\}\right) = \operatorname{Im}\frac{-i}{2}te^{it} = \frac{-1}{2}t\cos t$. The general solution is $y(t) = \frac{-1}{2}t\cos t + c_1\cos t + c_2\sin t$.

3. The functions $e^t\cos 2t$ and $e^t\sin 2t$ form a fundamental set. Let $y_p(t) = c_1e^t\cos 2t + c_2e^t\sin 2t$. Then the matrix equation $W(e^t\cos 2t, e^t\sin 2t)\begin{pmatrix}u_1'\\ u_2'\end{pmatrix} = \begin{pmatrix}0\\ e^t\end{pmatrix}$ implies that $u_1'(t) = \frac{-1}{2}\sin 2t$ and $u_2'(t) = \frac{1}{2}\cos 2t$. Hence, $u_1(t) = \frac{1}{4}cos2t$ and $u_2(t) = \frac{1}{4}\sin 2t$. From this we get $y_p(t) = \frac{1}{4}e^t\cos^2 2t + \frac{1}{4}e^t\sin^2 2t = \frac{1}{4}e^t$. On the other hand, the method of undetermined coefficients implies that a particular solution is of the form $y_p(t) = Ce^t$. Substitution gives $4Ce^t = e^t$ and hence $C = \frac{1}{4}$. It follows that $y_p(t) = \frac{1}{4}e^t$. Furthermore, the general solution is $y(t) = \frac{1}{4}e^t + c_1e^t\cos 2t + c_2e^t\sin 2t$.

5. A fundamental set is $\{e^t, e^{2t}\}$. The matrix equation $\begin{pmatrix}e^t & e^{2t}\\ e^t & 2e^{2t}\end{pmatrix}\begin{pmatrix}u_1'\\ u_2'\end{pmatrix} = \begin{pmatrix}0\\ e^{3t}\end{pmatrix}$ implies $u_1'(t) = -e^{2t}$ and $u_2'(t) = e^t$. Hence $u_1(t) = \frac{-1}{2}e^{2t}$, $u_2(t) = e^t$, and $y_p(t) = \frac{-1}{2}e^{2t}e^t + e^te^{2t} = \frac{1}{2}e^{3t}$. The general solution is $y(t) = \frac{1}{2}e^{3t} + c_1e^t + c_2e^{2t}$. The method of undetermined coefficients implies that a particular solution is of the form $y_p = Ce^{3t}$. Substitution gives $2Ce^{3t} = 3e^{3t}$ and hence $C = \frac{1}{2}$. The general solution is as above.

7. A fundamental set is $\{e^t, te^t\}$. The matrix equation $\begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{e^t}{t} \end{pmatrix}$ implies $u_1'(t) = -1$ and $u_2'(t) = \frac{1}{t}$. Hence, $u_1(t) = -t$, $u_2(t) = \ln t$, and $y_p(t) = -te^t + t\ln te^t$. Since $-te^t$ is a homogeneous solution we can write the general solution as $y(t) = t\ln te^t + c_1e^t + c_2te^t$.

9. The associated homogeneous equation is Cauchy-Euler with indicial equation $s^2-3s+2 = (s-2)(s-1)$. It follows that $\{t, t^2\}$ forms a fundamental set. We put the given equation is standard form to get $y'' - \frac{2}{t}y' + \frac{2}{t^2}y = t^2$. Thus $f(t) = t^2$. The matrix equation $\begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ t^2 \end{pmatrix}$ implies $u_1'(t) = -t^2$ and $u_2'(t) = t$. Hence $u_1(t) = \frac{-t^3}{3}$, $u_2(t) = \frac{t^2}{2}$, and $y_p(t) = \frac{-t^3}{3}t + \frac{t^2}{2}t^2 = \frac{t^4}{6}$. It follows that the general solution is $y(t) = \frac{t^4}{6} + c_1t + c_2t^2$.

11. The homogeneous equation is Cauchy-Euler with indicial equation $s^2 - 2s + 1 = (s-1)^2$. It follows that $\{t, t\ln t\}$ is a fundamental set. After writing in standard form we see the forcing function $f(t)$ is $\frac{1}{t}$. The matrix equation $\begin{pmatrix} t & t\ln t \\ 1 & \ln t + 1 \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{t} \end{pmatrix}$ implies $u_1'(t) = \frac{-\ln t}{t}$ and $u_2'(t) = \frac{1}{t}$. Hence $u_1(t) = \frac{-\ln^2 t}{2}$, $u_2(t) = \ln t$, and $y_p(t) = \frac{-t}{2}\ln^2 t + t\ln^2 t = \frac{t}{2}\ln^2 t$. The general solution is $y(t) = \frac{t}{2}\ln^2 t + c_1t + c_2t\ln t$.

13. The matrix equation $\begin{pmatrix} \tan t & \sec t \\ \sec^2 t & \sec t\tan t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix}$ implies $u_1'(t) = t$ and $u_2'(t) = -t\sin t$. Hence $u_1(t) = \frac{t^2}{2}$, $u_2(t) = t\cos t - \sin t$, and $y_p(t) = \frac{t^2}{2}\tan t + (t\cos t - \sin t)\sec t = \frac{t^2}{2}\tan t + t - \tan t$. Since $\tan t$ is a homogeneous solution we can write the general solution as $y(t) = \frac{t^2}{2}\tan t + t + c_1\tan t + c_2\sec t$.

15. After put in standard form the forcing function f is $4t^4$. The matrix equation $\begin{pmatrix} \cos t^2 & \sin t^2 \\ -2t\sin t^2 & 2t\cos 2t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 4t^4 \end{pmatrix}$ implies $u_1'(t) = -2t^3\sin t^2$ and $u_2'(t) = 2t^3\cos t^2$. Integration by parts gives $u_1(t) = t^2\cos t^2 - \sin t^2$ and $u_2(t) = t^2\sin t^2 + \cos t^2$. Hence $y_p(t) = t^2\cos^2 t^2 - \cos t^2\sin t^2 + t^2\sin^2 t^2 + \cos t^2\sin t^2 = t^2$. The general solution is $y(t) = t^2 + c_1\cos t^2 + c_2\sin t^2$.

17. Let a and t be in the interval I. Let z_1 and z_2 be the definite integrals defined as follows:

$$\begin{aligned} z_1(t) &= \int_a^t \frac{-y_2(x)f(x)}{w(y_1,y_2)(x)}\,dx \\ z_2(t) &= \int_a^t \frac{y_1(x)f(x)}{w(y_1,y_2)(x)}\,dx. \end{aligned}$$

These definite integrals determine the constant of integration in Theorem 1 so that $z_1(a) = z_2(a) = 0$. It follows that

$$\begin{aligned} y_p(t) &= z_1(t)y_1(t) + z_2(t) + y_2(t) \\ &= \int_a^t \frac{-y_2(x)y_1(t)f(x)}{w(y_1,y_2)(x)}\,dx + \int_a^t \frac{y_1(x)y_2(t)f(x)}{w(y_1,y_2)(x)} \\ &= \int_a^t \frac{(y_1(x)y_2(t) - y_2(x)y_1(t))}{w(y_1,y_2)(x)} f(x)\,dx \\ &= \int_a^t \frac{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} f(x)\,dx. \end{aligned}$$

19. Let $y_1(t) = e^{-at}$ and $y_2(t) = e^{at}$. Then $\{y_1, y_2\}$ is a fundamental set. We have

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix} = e^{-ax}e^{at} - e^{ax}e^{-at} = e^{a(t-x)} - e^{-a(t-x)} = 2\sinh(a(t-x))$$

and

$$w(y_1,y_2)(x) = \begin{vmatrix} e^{-at} & e^{at} \\ -ae^{-at} & ae^{at} \end{vmatrix} = 2a.$$

Thus

$$\begin{aligned} y_p(t) &= \int_0^t \frac{2\sinh a(t-x)}{2a} f(x)\,dx \\ &= \frac{1}{a} f(t) * \sinh at. \end{aligned}$$

Applying the Laplace transform to $y'' - a^2y = f$, with initial conditions $y(0) = y'(0) = 0$, gives $s^2Y(s) - a^2Y(s) = F(s)$. Solving for $Y(s)$ we get

$$Y(s) = \frac{F(s)}{s^2 - a^2} = \frac{1}{a}\frac{a}{s^2 - a^2}F(s).$$

The convolution theorem gives a particular solution

$$y_p(t) = \frac{1}{a}\sinh at * f(t).$$

21. Let $y_1(t) = e^{at}$ and $y_2(t) = e^{bt}$. Then $\{y_1, y_2\}$ is a fundamental set. We have

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix} = e^{ax}e^{bt} - e^{bx}e^{at} = e^{ax+bt} - e^{bx+at}$$

and

$$w(y_1, y_2)(x) = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix} = (b-a)e^{(a+b)x}.$$

Thus

$$\begin{aligned} y_p(t) &= \int_0^t \frac{e^{ax+bt} - e^{bx+at}}{(b-a)e^{(a+b)x}} f(x)\,dx \\ &= \frac{1}{b-a}\int_0^t (e^{b(t-x)} - e^{a(t-x)})f(x)\,dx \\ &= \frac{1}{b-a} f(t) * (e^{bt} - e^{at}). \end{aligned}$$

Applying the Laplace transform to $y'' - (a+b)y' + aby = f$, with initial conditions $y(0) = y'(0) = 0$, gives $s^2Y(s) - (a+b)sY(s) + abY(s) = F(s)$. Solving for $Y(s)$ we get

$$Y(s) = \frac{F(s)}{(s-a)(s-b)} = \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right)F(s).$$

The convolution theorem gives a particular solution

$$y_p(t) = \frac{1}{a-b}(e^{at} - e^{bt}) * f(t).$$

Section 6.1

1. Graph (c)

3. Graph (e)

5. Graph (f)

7. Graph (h)

9. $\int_0^5 f(t)\,dt = \int_0^2 (t^2-4)\,dt + \int_2^3 0\,dt + \int_3^5 (-t+3)\,dt = \left(t^3/3 - 4t\right)\Big|_0^2 + 0 + \left(-t^2/2 + 3t\right)\Big|_3^5 = (8/3 - 8) + (-25/2 + 15) - (-9/2 + 9) = -22/3.$

11. $\int_0^{2\pi} |\sin x|\ dx = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} -\sin x\,dx = -\cos x|_0^{\pi} + \cos x|_{\pi}^{2\pi} = 4.$

13. $\int_2^5 f(t)\,dt = \int_2^3 (3-t)\,dt + \int_3^4 2(t-3)\,dt + \int_4^6 2\,dt = 1/2 + 1 + 4 = 11/2$

15. $\int_0^6 f(u)\,du = \int_0^1 u\,du + \int_1^2 (2-u)\,du + \int_2^6 1\,du = 1/2 + 1/2 + 4 = 5.$

17. A is true since $y(t)$ satisfies the differential equation on each subinterval. B is true since the left and right limits agree at $t = 2$. C is not true since $y(0) = 1 \neq 2$.

19. A is true since $y(t)$ satisfies the differential equation on each subinterval. B is false since $\lim_{t\to 2^-} y(t) = 1 + e^{-8}$ while $\lim_{t\to 2^+} y(t) = 1$. C is false since B is false.

21. A is true since $y(t)$ satisfies the differential equation on each subinterval. B is true since $\lim_{t\to 1^-} y(t) = -2e + e^2 = \lim_{t\to 1^+} y(t)$. C is false since $\lim_{t\to 1^-} y'(t) = -3e + 2e^2$ while $\lim_{t\to 1^+} y'(t) = 3e^2 - 2e$. D is false since C is false.

23. A is true since $y(t)$ satisfies the differential equation on each subinterval. B is true since $\lim_{t\to 1^-} y(t) = -2e + e^2 = \lim_{t\to 1^+} y(t)$. C is true since $\lim_{t\to 1^-} y'(t) = -3e + 2e^2 = \lim_{t\to 1^+} y'(t)$. D is true since $y(0) = y'(0) = 0$.

25. The general solution of $y' - y = 1$ on the interval $[0, 2)$ is found by using the integrating factor e^{-t}. The general solution is $y(t) = -1 + ce^t$ and the initial condition $y(0) = 0$ gives $c = 1$, so that $y(t) = -1 + e^t$ for $t \in [0, 2)$. Continuity of $y(t)$ at $t = 2$ will then give $y(2) = \lim_{t\to 2^-} y(t) = -1 + e^2$, which will provide the initial condition for the next interval $[2, 4)$. The general solution of $y' - y = -1$ on $[2, 4)$ is $y(t) = 1 + ke^t$. Thus $-1 + e^2 = y(2) = 1 + ke^2$ and solve for k to get $k = -2e^{-2} + 1$, so that $y(t) = 1 + (-2e^{-2} + 1)e^t$ for $t \in [2, 4)$. Continuity will then give $y(4) = 1 + (-2e^{-2} + 1)e^4$, which will provide the initial condition for the next interval $[4, \infty)$. The general solution to $y' - y = 0$ on $[4, \infty)$ is $y(t) = be^t$ and the constant b is obtained from the initial condition $be^4 = y(4) = 1 + (-2e^{-2} + 1)e^4$, which gives $b = e^{-4} - 2e^{-2} + 1$, so that $y(t) = (e^{-4} - 2e^{-2} + 1)e^t$ for $t \in [4, \infty)$. Putting these three pieces together, we find that the solution is

$$y(t) = \begin{cases} -1 + e^t & \text{if } 0 \le t < 2, \\ 1 - 2e^{t-2} + e^t & \text{if } 2 \le t < 4 \\ e^{t-4} - 2e^{t-2} + e^t & \text{if } 4 \le t < \infty. \end{cases}$$

27. The general solution of $y' - y = f(t)$ on any interval is found by using the integrating factor e^{-t}. The general solution on the interval $[0, 1)$ is $y(t) = ae^t$ and since the initial condition is $y(0) = 0$, the solution on $[0, 1)$ is $y(t) = 0$. Continuity then given $y(1) = 0$, which will be the initial condition for the interval [1, 2). The general solution of $y' - y = t - 1$ on the interval [1, 2) is $y(t) = -t + be^t$ and the initial condition $y(1) = 0$ gives $0 = -1 + be^1$ so that $b = e^{-1}$. Thus $y(t) = -t + e^{-1}e^t = -t + e^{t-1}$ for $t \in [0, 2)$. Continuity of $y(t)$ at $t = 2$ will then give $y(2) = \lim_{t\to 2^-} y(t) = -2 + e^1$, which will provide the initial condition for the next interval [2, 3). The general solution of $y' - y = 3 - t$ on [2, 3) is $y(t) = t - 2 + ce^t$. Thus $-2 + e^1 = y(2) = ce^2$ and solve for c to get $c = -2e^{-2} + e^{-1}$, so that $y(t) = t - 2 + (-2e^{-2} + e^{-1})e^t = t - 2 - 2e^{t-2} + e^{t-1}$ for $t \in [2, 3)$. Continuity will then give $y(3) = 1 - 2e^1 + e^2$, which will provide the initial condition for the next interval $[3, \infty)$. The general solution to $y' - y = 0$ on $[4, \infty)$ is $y(t) = ke^t$ and the constant k is obtained from the initial condition $ke^3 = y(3) = 1 - 2e^1 + e^2$, which gives $c = e^{-3} - 2e^{-2} + e^{-1}$, so that $y(t) = (e^{-3} - 2e^{-2} + e^{-1})e^t = e^{t-3} - 2e^{t-2} + e^{t-1}$ for $t \in [3, \infty)$. Putting these three pieces together, we find that the solution is

$$y(t) = \begin{cases} 0 & \text{if } 0 \le t < 1, \\ -t + e^{t-1} & \text{if } 1 \le t < 2, \\ t - 2 - 2e^{t-2} + e^{t-1} & \text{if } 2 \le t < 3 \\ e^{t-3} - 2e^{t-2} + e^{t-1} & \text{if } 3 \le t < \infty. \end{cases}$$

29. The characteristic polynomial of the equation $y'' - y = f(t)$ is $s^2 - 1 = (s-1)(s+1)$ so the homogeneous equation has the solution $y_h(t) = ae^t + be^{-t}$ for constants a and b. On the interval [0, 1] the equation $y'' - y = t$ has a particular solution $y_p(t) = -t$ so the general solution has the form $y(t) = -t + ae^t + be^{-t}$. The initial conditions give $0 = y(0) = a + b$ and $1 = y'(0) = -1 + a - b$. Solving gives $a = 1$, $b = -1$ so $y(t) = -t + e^t - e^{-t}$ on [0, 1). By continuity it follows that $y(1) = -1 + e^1 - e^{-1}$ and $y'(1) = -1 + e^1 + e^{-1}$ and these constitute the initial values for the equation $y'' - y = 0$ on the interval $[1, \infty)$. The general solution on this interval is $y(t) = ae^t + be^{-t}$ and at $t = 1$ we get $y(1) = ae^1 + be^{-1} = -1 + e^1 - e^{-1}$ and $y'(1) = ae^1 - be^{-1} = -1 + e^1 + e^{-1}$. Solving for a and b gives $a = 1 - e^{-1}$ and $b = -1$ so that $y(t) = (1 - e^{-1})e^t - e^{-t} = e^t - e^{t-1} - e^{-1}$. Putting the two pieces together gives

$$y(t) = \begin{cases} -t + e^t - e^{-t} & \text{if } 0 \le t < 1, \\ e^t - e^{t-1} - e^{-1} & 1 \le t < \infty. \end{cases}$$

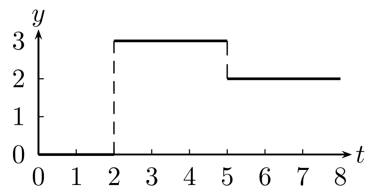
33. 1. $|f(t)| = |\sin(1/t)| \le 1$ for all $t \ne 0$, while $|f(0)| = |0| = 0 \le 1$.
2. It is enough to observe that $\lim_{t\to 0^+}$ does not exist. But letting $t_n = \frac{1}{n\pi}$ gives $f(t_n) = \sin n\pi = 0$ for all positive integers n, while letting $t_n = \frac{2}{(4n+1)\pi}$ gives $f(t_n) = \sin(1/t_n) = \sin((4n+1)\pi/2) = \sin(2n\pi +$

$\frac{\pi}{2}) = 1$ so there is one sequence $t_n \to 0$ with $f(t_n) \to 0$ while another sequence $t_n \to 0$ with $f(t_n) \to 1$ so $f(t)$ cannot be continuous at 0.

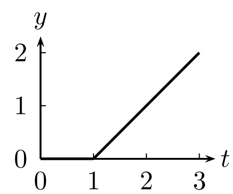
3. To be piecewise continuous, $f(t)$ would have to have a limit at t approaches 0 from above, and this is not true as shown in part 2.

Section 6.2

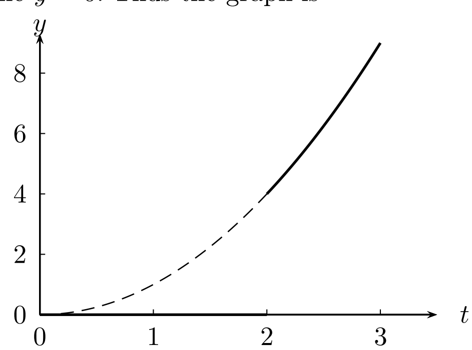
1. $f(t) = 3h(t-2) - h(t-5) = \begin{cases} 0 & \text{if } t < 2, \\ 3 & \text{if } 2 \le t < 5, \\ 2 & \text{if } t \ge 5. \end{cases}$ Thus, the graph is



3. This function is $g(t-1)h(t-1)$ where $g(t) = t$, so the graph of $f(t)$ is the graph of $g(t) = t$ translated 1 unit to the right and then truncated at $t = 1$, with the graph before $t = 1$ replaced by the line $y = 0$. Thus the graph is

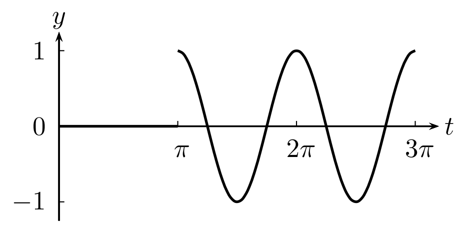


5. This function is just t^2 truncated at $t = 2$, with the graph before $t = 2$ replaced by the line $y = 0$. Thus the graph is



where the dashed line is the part of the t^2 graph that has been truncated. It is only shown for emphasis and it is not part of the graph.

7. This is the function $\cos 2t$ shifted π units to the right and then truncated at $t = \pi$. The graph is



9. (a) $(t-2)\chi_{[2,\infty)}(t)$; (b) $(t-2)h(t-2)$;
(c) $\mathcal{L}\{(t-2)h(t-2)\} = e^{-2s}\mathcal{L}\{t\} = e^{-2s}/s^2$.

11. (a) $(t+2)\chi_{[2,\infty)}(t)$; (b) $(t+2)h(t-2)$;
(c) $\mathcal{L}\{(t+2)h(t-2)\} = e^{-2s}\mathcal{L}\{(t+2)+2\} = e^{-2s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$.

13. (a) $t^2\chi_{[4,\infty)}(t)$; (b) $t^2h(t-4)$;
(c) $\mathcal{L}\left\{t^2h(t-4)\right\} = e^{-4s}\mathcal{L}\left\{(t+4)^2\right\} = e^{-4s}\mathcal{L}\left\{t^2+8t+16\right\}$
$= e^{-4s}\left(\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right)$.

15. (a) $(t-4)^2\chi_{[2,\infty)}(t)$; (b) $(t-4)^2h(t-2)$;
(c) $\mathcal{L}\left\{(t-4)^2h(t-2)\right\} = e^{-2s}\mathcal{L}\left\{((t+2)-4)^2\right\} = e^{-2s}\mathcal{L}\left\{t^2-4t+4\right\}$
$= e^{-2s}\left(\frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}\right)$.

17. (a) $e^t\chi_{[4,\infty)}(t)$; (b) $e^th(t-4)$;
(c) $\mathcal{L}\left\{e^th(t-4)\right\} = e^{-4s}\mathcal{L}\left\{e^{t+4}\right\} = e^{-4s}e^4\mathcal{L}\left\{e^t\right\}$
$= e^{-4(s-1)}\frac{1}{s-1}$.

19. (a) $te^t\chi_{[4,\infty)}(t)$; (b) $te^th(t-4)$;
(c) $\mathcal{L}\left\{te^th(t-4)\right\} = e^{-4s}\mathcal{L}\left\{(t+4)e^{t+4}\right\} = e^{-4s}e^4\mathcal{L}\left\{te^t + 4e^t\right\}$
$= e^{-4(s-1)}\left(\frac{1}{(s-1)^2} + \frac{4}{s-1}\right)$.

21. (a) $t\chi_{[0,1)}(t) + (2-t)\chi_{[1,\infty)}(t)$; (b) $t + (2-2t)h(t-1)$;
(c) $\mathcal{L}\{t + (2-2t)h(t-1)\} = \mathcal{L}\{t\} + e^{-s}\mathcal{L}\{(2-2(t+1))\}$
$= \mathcal{L}\{t\} + e^{-s}\mathcal{L}\{-2t\} = \frac{1}{s^2} - \frac{2e^{-s}}{s^2}$.

23. (a) $t^2\chi_{[0,2)}(t) + 4\chi_{[2,3)}(t) + (7-t)\chi_{[3,\infty)}(t)$;
(b) $t^2 + (4-t^2)h(t-2) + (3-t)h(t-3)$;
(c) $\mathcal{L}\left\{t^2 + (4-t^2)h(t-2) + (3-t)h(t-3)\right\}$

$= \mathcal{L}\left\{t^2\right\} + e^{-2s}\mathcal{L}\left\{4-(t+2)^2\right\} + e^{-3s}\mathcal{L}\left\{3-(t+3)\right\}$
$= \dfrac{2}{s^3} - e^{-2s}\left(\dfrac{2}{s^3}+\dfrac{4}{s^2}\right) - \dfrac{e^{-3s}}{s^2}.$

25. (a) $\sum_{n=0}^{\infty}(t-n)\chi_{[n,n+1)}(t)$;
(b) $t - \sum_{n=1}^{\infty} h(t-n)$;
(c) $\mathcal{L}\left\{t - \sum_{n=1}^{\infty} h(t-n)\right\} = \mathcal{L}\left\{t\right\} - \sum_{n=1}^{\infty} \mathcal{L}\left\{h(t-n)\right\}$
$= \dfrac{1}{s^2} - \sum_{n=1}^{\infty} \dfrac{e^{-ns}}{s} = \dfrac{1}{s^2} - \dfrac{1}{s}\sum_{n=1}^{\infty}\left(e^{-s}\right)^n$
$= \dfrac{1}{s^2} - \dfrac{e^{-s}}{s(1-e^{-s})}.$

27. (a) $\sum_{n=0}^{\infty}(2n+1-t)\chi_{[2n,2n+2)}(t)$; (b) $-(t+1)+2\sum_{n=0}^{\infty} h(t-2n)$;
(c) $-\dfrac{1}{s^2} - \dfrac{1}{s} + \dfrac{2}{s(1-e^{-2s})}.$

29. $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2}\right\} = h(t-3)\,\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\}\Big|_{t\to t-3}$
$= h(t-3)\,(t)|_{t\to t-3} = (t-3)h(t-3) = \begin{cases} 0 & \text{if } 0 \le t < 3, \\ t-3 & \text{if } t \ge 3. \end{cases}$

31. $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\} = h(t-\pi)\,\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}\Big|_{t\to t-\pi}$
$= h(t-\pi)\,(\sin t)|_{t\to t-\pi} = h(t-\pi)\sin(t-\pi)$
$= \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ \sin(t-\pi) & \text{if } t \ge \pi \end{cases} = \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ -\sin t & \text{if } t \ge \pi. \end{cases}$

33. $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+2s+5}\right\} = h(t-\pi)\,\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+5}\right\}\Big|_{t\to t-\pi}$
$= h(t-\pi)\,\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2+2^2}\right\}\Big|_{t\to t-\pi} = h(t-\pi)\,\left(\tfrac{1}{2}e^{-t}\sin 2t\right)\Big|_{t\to t-\pi}$
$= \tfrac{1}{2}e^{-(t-\pi)}\sin 2(t-\pi)h(t-\pi) = \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ \tfrac{1}{2}e^{-(t-\pi)}\sin 2t & \text{if } t \ge \pi. \end{cases}$

35. $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2+4}\right\} = h(t-2)\,\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\}\Big|_{t\to t-2}$
$= h(t-2)\,\left(\tfrac{1}{2}\sin 2t\right)\Big|_{t\to t-2} = \tfrac{1}{2}h(t-2)\sin 2(t-2)$
$= \begin{cases} 0 & \text{if } 0 \le t < 2, \\ \tfrac{1}{2}\sin 2(t-2) & \text{if } t \ge 2. \end{cases}$

37. $\mathcal{L}^{-1}\left\{\frac{se^{-4s}}{s^2+3s+2}\right\} = h(t-4)\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+3s+2}\right\}\Big|_{t\to t-4}$
$= h(t-4)\,\mathcal{L}^{-1}\left\{\frac{2}{s+2} - \frac{1}{s+1}\right\}\Big|_{t\to t-4}$
$= h(t-4)\,(2e^{-2t} - e^{-t})\big|_{t\to t-4} = h(t-4)\left(2e^{-2(t-4)} - e^{-(t-4)}\right)$
$= \begin{cases} 0 & \text{if } 0 \le t < 4, \\ 2e^{-2(t-4)} - e^{-(t-4)} & \text{if } t \ge 4. \end{cases}$

39. $\mathcal{L}^{-1}\left\{\frac{1-e^{-5s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - h(t-5)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\Big|_{t\to t-5}$
$= t - h(t-5)\,(t)|_{t\to t-5} = t - (t-5)h(t-5) = \begin{cases} t & \text{if } 0 \le t < 5, \\ 5 & \text{if } t \ge 5. \end{cases}$

41. $\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{2s+1}{s^2+6s+13}\right\} = h(t-\pi)\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+6s+13}\right\}\Big|_{t\to t-\pi}$
$= h(t-\pi)\,\mathcal{L}^{-1}\left\{\frac{2(s+3)-5}{(s+3)^2+2^2}\right\}\Big|_{t\to t-\pi}$
$= h(t-\pi)\,\mathcal{L}^{-1}\left\{\frac{2(s+3)}{(s+3)^2+2^2}\right\}\Big|_{t\to t-\pi}$
$+ h(t-\pi)\,\mathcal{L}^{-1}\left\{\frac{-5}{(s+3)^2+2^2}\right\}\Big|_{t\to t-\pi}$
$= h(t-\pi)\,(2e^{-3t}\cos 2t - \frac{5}{2}e^{-3t}\sin 2t)\big|_{t\to t-\pi}$
$= h(t-\pi)e^{-3(t-\pi)}\left(2\cos 2(t-\pi) - \frac{5}{2}\sin 2(t-\pi)\right)$
$= \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ e^{-3(t-\pi)}\left(2\cos 2t - \frac{5}{2}\sin 2t\right) & \text{if } t \ge \pi. \end{cases}$

43. Let $b > 0$. Since f_1 and f_2 are piecewise continuous on $[0,\infty)$ they only have finitely many jump discontinuities on $[0,b)$. It follows that f_1+cf_2 have only finitely many jump on $[0,b)$. Thus f_1+cf_2 is piecewise continuous on $[0,\infty)$.

Section 6.3

1. We write the forcing function as $f(t) = 3h(t-1)$. Applying the Laplace transform, partial fractions, and simplifying gives

$$Y(s) = \frac{-3}{s(s+2)}e^{-s} = \frac{-3}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)e^{-s}.$$

Laplace inversion now gives

$$y = -\frac{3}{2}h(t-1)\left(1-e^{-2(t-1)}\right) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ -\frac{3}{2}\left(1-e^{-2(t-1)}\right) & \text{if } 1 \le t < \infty \end{cases}.$$

3. We write the forcing function as $f(t) = 2\chi_{[2,3)} = 2h(t-2) - 2h(t-3)$. Applying the Laplace transform, partial fractions, and simplifying gives

$$\begin{aligned} Y(s) &= \frac{2}{s(s-3)}\left(e^{-2s} - e^{-3s}\right) \\ &= \frac{2}{3}\left(\frac{1}{s-3} - \frac{1}{s}\right)\left(e^{-2s} - e^{-3s}\right). \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned} y &= \frac{2}{3}\left(\left(e^{3(t-2)} - 1\right)h(t-2) - \left(e^{3(t-3)} - 1\right)h(t-3)\right) \\ &= \begin{cases} 0 & \text{if } 0 \le t < 2 \\ \frac{2}{3}\left(e^{3(t-2)} - 1\right) & \text{if } 2 \le t < 3 \\ \frac{2}{3}\left(e^{3(t-2)} - e^{3(t-3)}\right) & \text{if } 3 \le t < \infty \end{cases}. \end{aligned}$$

5. We write the forcing function as

$$\begin{aligned} f(t) &= 12e^t\chi_{[0,1)} + 12e\chi_{[1,\infty)} \\ &= 12e^t - 12(e^t - e)h(t-1). \end{aligned}$$

Applying the Laplace transform, partial fractions, and simplifying gives

$$\begin{aligned} Y(s) &= \frac{2}{s-4} + \frac{12}{(s-1)(s-4)} - e^{-s}\left(\frac{12e}{(s-1)(s-4)} - \frac{12e}{s(s+4)}\right) \\ &= \frac{6}{s-4} - \frac{4}{s-1} - e^{-s}e\left(\frac{-4}{s-1} + \frac{1}{s-4} + \frac{3}{s}\right). \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned} y &= 6e^{4t} - 4e^t - e\left(-4e^{t-1} + e^{4(t-1)} + 3\right)h(t-1) \\ &= 6e^{4t} - 4e^t + 4e^t h(t-1) - e^{4t-3}h(t-1) - 3eh(t-1) \\ &= \begin{cases} 6e^{4t} - 4e^t & \text{if } 0 \le t < 1 \\ 6e^{4t} - e^{4t-3} - 3e & \text{if } 1 \le t < \infty \end{cases}. \end{aligned}$$

7. Applying the Laplace transform, partial fractions, and simplifying gives

$$\begin{aligned} Y(s) &= \frac{e^{-3s}}{s(s^2+9)} \\ &= \frac{1}{9}\left(\frac{1}{s} - \frac{s}{s^2+9}\right) e^{-3s}. \end{aligned}$$

Laplace inversion now gives

$$y = \frac{1}{9}(1-\cos 3(t-3))h(t-3) = \begin{cases} 0 & \text{if } 0 \le t < 3 \\ \frac{1}{9}(1-\cos 3(t-3)) & \text{if } 3 \le t < \infty \end{cases}.$$

9. Write the forcing function as $f(t) = 6\chi_{[1,3)} = 6h(t-1) - 6h(t-3)$. Now apply the Laplace transform, partial fractions, and simplify to get

$$\begin{aligned} Y(s) &= \frac{6}{s(s+2)(s+3)}\left(e^{-s} - e^{-3s}\right) \\ &= \left(\frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3}\right)\left(e^{-s} - e^{-3s}\right) \end{aligned}$$

Now we take the inverse Laplace transform and simplify to get

$$\begin{aligned} y &= \left(1 - 3e^{-2(t-1)} + 2e^{-3(t-1)}\right) h(t-1) \\ &\quad - \left(1 - 3e^{-2(t-3)} + 2e^{-3(t-3)}\right) h(t-3) \\ &= \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 1 - 3e^{-2(t-1)} + 2e^{-3(t-1)} & \text{if } 1 \le t < 3 \\ 3e^{-2(t-3)} - 3e^{-2(t-1)} - 2e^{-3(t-3)} + 2e^{-3(t-1)} & \text{if } 3 \le t < \infty \end{cases} \end{aligned}$$

11. Apply the Laplace transform, partial fractions, and simplify to get

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)^2} + \frac{1}{s(s+1)^2} e^{-3s} \\ &= \frac{1}{(s+1)^2} + \left(-\frac{1}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s}\right) e^{-3s} \end{aligned}$$

Laplace inversion gives

$$\begin{aligned} y &= te^{-t} + \left(-(t-3)e^{-(t-3)} - e^{-(t-3)} + 1\right) h(t-3) \\ &= te^{-t} + \left(1 - (t-2)e^{-(t-3)}\right) h(t-3) \\ &= \begin{cases} te^{-t} & \text{if } 0 \le t < 3 \\ 1 + te^{-t} - (t-2)e^{-(t-3)} & \text{if } 3 \le t < \infty \end{cases}. \end{aligned}$$

13. For the first three minutes, source one adds salt at a rate of $1\frac{\text{lb}}{\text{gal}} \cdot 2\frac{\text{gal}}{\text{min}} = 2\frac{\text{lbs}}{\text{min}}$. and after that source two takes over and adds salt at a rate of $5\frac{\text{lb}}{\text{gal}} \cdot 2\frac{\text{gal}}{\text{min}} = 10\frac{\text{lbs}}{\text{min}}$. Thus the rate at which salt is being added is given by the function

$$\begin{aligned} f(t) &= \begin{cases} 2 & \text{if } 0 \le t < 3 \\ 10 & \text{if } 3 \le t < \infty. \end{cases} \\ &= 2\chi_{[0,3)} + 10\chi_{[3,\infty)} \\ &= 2(1 - h(t-3)) + 10(h(t-3)) \\ &= 2 + 8h(t-3). \end{aligned}$$

The output rate of salt is given by $\frac{y(t)}{4} \cdot 2 = \frac{y(t)}{2}$ lbs/min. We are thus led to the differential equation

$$y' + \frac{1}{2}y(t) = 2 + 8h(t-3), \quad y(0) = 0.$$

We take the Laplace transform of both sides and use partial fractions to get

$$\begin{aligned} Y(s) &= \frac{2}{s(s+1/2)} + \frac{8e^{-3s}}{s(s+1/2)} \\ &= \frac{4}{s} - \frac{4}{s+1/2} + e^{-3s}\left(\frac{16}{s} - \frac{16}{s+1/2}\right). \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned} y(t) &= 4 - 4e^{\frac{-t}{2}} + 16h(t-3) - 16e^{\frac{-(t-3)}{2}}h(t-3) \\ &= \begin{cases} 4 - 4e^{\frac{-t}{2}} & \text{if } 0 \le t < 3 \\ 20 - 4e^{\frac{-t}{2}} - 16e^{\frac{-(t-3)}{2}} & \text{if } t \ge 3. \end{cases} \end{aligned}$$

15. For the first two minutes, source one adds salt at a rate of $1\frac{\text{kg}}{\text{L}} \cdot 3\frac{\text{L}}{\text{min}} = 3\frac{\text{kg}}{\text{min}}$. Thereafter source two takes over for two minutes but the input rate of salt is 0. Thereafter source on take over again and adds salt to the tank at a rate of $3\frac{\text{kg}}{\text{min}}$. Thus the rate at which salt is being added is given by the function

$$\begin{aligned} f(t) &= \begin{cases} 3 & \text{if } 0 \le t < 2 \\ 0 & \text{if } 2 \le t < 4 \\ 3 & \text{if } 4 \le t < \infty. \end{cases} \\ &= 3\chi_{[0,2)} + 3\chi_{[4,\infty)} \\ &= 3(1 - h(t-2) + h(t-4)). \end{aligned}$$

The output rate of salt is given by $\frac{y(t)}{10} \cdot 3 = \frac{3}{10}y(t)$ kg/min. We are thus led to the differential equation

$$y' + \frac{3}{10}y(t) = 3(1 - h(t-2) + h(t-4)), \quad y(0) = 2.$$

We take the Laplace transform of both sides, simplify, and use partial fractions to get

$$\begin{aligned} Y(s) &= \frac{2}{s + 3/10} + \frac{3}{s(s + 3/10)}(1 - e^{-2s} + e^{-4s}) \\ &= \frac{10}{s} - \frac{8}{s + 3/10} + \left(\frac{10}{s} - \frac{10}{s + 3/10}\right)(e^{-4s} - e^{-2s}). \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned} y(t) &= 10 - 8e^{-3t/10} - \left(10 - 10e^{-3(t-2)/10}\right) h(t-2) \\ &\qquad + \left(10 - 10e^{-3(t-4)/10}\right) h(t-4) \\ &= \begin{cases} 10 - 8e^{-3t/10} & \text{if } 0 \le t < 2 \\ 10e^{-3(t-2)/10} - 8e^{-3t/10} & \text{if } 2 \le t < 4 \\ 10 - 8e^{-3t/10} + 10e^{-3(t-2)/10} - 10e^{-3(t-4)/10} & \text{if } 4 \le t < \infty \end{cases}. \end{aligned}$$

Section 6.4

1. Take the Laplace transform, solve for $Y(s)$, and simplify to get $Y(s) = \frac{e^{-s}}{s+2}$. Laplace inversion then gives

$$\begin{aligned} y &= e^{-2(t-1)}h(t-1) \\ &= \begin{cases} 0 & \text{if } 0 \le t < 1 \\ e^{-2(t-1)} & \text{if } 1 \le t < \infty \end{cases}. \end{aligned}$$

3. Take the Laplace transform, solve for $Y(s)$, and simplify to get $Y(s) = \frac{2}{s-4} + \frac{e^{-4s}}{s-4}$. Laplace inversion then gives

$$\begin{aligned} y &= 2e^{4t} + e^{4(t-4)}h(t-4) \\ &= \begin{cases} 2e^{4t} & \text{if } 0 \le t < 4 \\ 2e^{4t} + e^{4(t-4)} & \text{if } 4 \le t < \infty \end{cases}. \end{aligned}$$

5. We begin by taking the Laplace transform of each side and simplifying to get $Y(s) = \frac{1}{s^2+4} + \frac{e^{-\pi s}}{s^2+4}$. Laplace inversion then gives

$$\begin{aligned} y &= \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2(t-\pi)h(t-\pi) \\ &= \frac{1}{2}\sin 2t + \frac{1}{2}\sin(2t)\, h(t-\pi) \\ &= \begin{cases} \frac{\sin 2t}{2} & \text{if } 0 \le t < \pi \\ \sin 2t & \text{if } \pi \le t < \infty \end{cases}. \end{aligned}$$

7. Apply the Laplace transform, partial fractions, and simplify to get

$$\begin{aligned} Y(s) &= \frac{s+3}{(s+1)(s+3)} + \frac{2e^{-2s}}{(s+1)(s+3)} \\ &= \frac{1}{s+1} + \left(\frac{1}{s+1} - \frac{1}{s+3}\right)e^{-2s}. \end{aligned}$$

Laplace inversion gives

$$\begin{aligned} y &= e^{-t} + \left(e^{-(t-2)} - e^{-3(t-2)}\right) h(t-2) \\ &= \begin{cases} e^{-t} & \text{if } \ 0 \le t < 2 \\ e^{-t} + e^{-(t-2)} - e^{-3(t-2)} & \text{if } \ 2 \le t < \infty \end{cases}. \end{aligned}$$

9. Take the Laplace transform, apply partial fractions, and simplify to get

$$\begin{aligned} Y(s) &= -\frac{s+1}{(s+2)^2} + \frac{3}{(s+2)^2}e^{-s} \\ &= \frac{1}{(s+2)^2} - \frac{1}{s+2} + \frac{3}{(s+2)^2}e^{-s}. \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned} y &= te^{-2t} - e^{-2t} + 3(t-1)e^{-2(t-1)}h(t-1) \\ &= \begin{cases} te^{-2t} - e^{-2t} & \text{if } \ 0 \le t < 1 \\ te^{-2t} - e^{-2t} + 3(t-1)e^{-2(t-1)} & \text{if } \ 1 \le t < \infty \end{cases}. \end{aligned}$$

11. The input rate of salt is $6 + 4\delta_3$ while the output rate is $3\frac{y(t)}{12}$. We thus have the differential equation $y' + \frac{1}{4}y = 6 + 4\delta_3, \quad y(0) = 0$. Take the Laplace transform, apply partial fractions, and simplify to get

$$
\begin{aligned}
Y(s) &= \frac{6}{s(s+1/4)} + \frac{4}{s+1/4}e^{-3s} \\
&= \frac{24}{s} - \frac{24}{s+1/4} + \frac{4}{s+1/4}e^{-3s}.
\end{aligned}
$$

Laplace inversion now gives

$$
\begin{aligned}
y &= 24 - 24e^{-\frac{1}{4}t} + 4e^{-\frac{1}{4}(t-3)}h(t-3) \\
&= \begin{cases} 24 - 24e^{-\frac{1}{4}t} & \text{if } 0 \le t < 3 \\ 24 - 24e^{-\frac{1}{4}t} + 4e^{-\frac{1}{4}(t-3)} & \text{if } 3 \le t < \infty \end{cases}.
\end{aligned}
$$

13. Clearly, $y(0) = 0$. The input rate is $\delta_0 + \delta_2 + \delta_4 + \delta_6$ while the output rate is y. We are thus led to the differential equation $y' + y = \sum_{k=0}^{3} \delta_{2k}, \quad y(0) = 0$. Take the Laplace transform and solve for $Y(s)$ to get

$$
Y(s) = \sum_{k=0}^{3} \frac{e^{-2ks}}{s+1}.
$$

Laplace inversion gives

$$
\begin{aligned}
y &= \sum_{k=0}^{3} e^{-(t-2k)}h(t-2k) \\
&= \begin{cases} e^{-t} & \text{if } 0 \le t < 2 \\ e^{-t} + e^{-(t-2)} & \text{if } 2 \le t < 4 \\ e^{-t} + e^{-(t-2)} + e^{-(t-4)} & \text{if } 4 \le t < 6 \\ e^{-t} + e^{-(t-2)} + e^{-(t-4)} + e^{-(t-6)} & \text{if } 6 \le t < \infty \end{cases}.
\end{aligned}
$$

Using the formula $1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$ we get

$$
\begin{aligned}
y(6) &= \sum_{k=0}^{3} e^{-(6-2k)} = \sum_{k=0}^{3} e^{-2k} = \sum_{k=0}^{3} (e^{-2})^k \\
&= \frac{1 - \left(e^{-2}\right)^4}{1 - e^{-2}} = 1.156 \text{ lb.}
\end{aligned}
$$

15. The mass is $m = 2$. The spring constant k is given by $k = 8/1 = 8$. The damping constant is given by $\mu = 8/1 = 8$. The external force is $2\delta_4$. The initial conditions are $y(0) = .1$ and $y'(0) = .05$. The equation $2y'' + 8y' + 8y = 2\delta_4, \; y(0) = .1, \; y'(0) = .05$ models the motion of the body. Divide by two to get $y'' + 4y' + 4y = \delta_4$. Apply the Laplace transform to get

$$Y(s) = \frac{.1}{s+2} + \frac{1}{(s+2)^2}e^{-4s}.$$

Laplace inversion now gives

$$\begin{aligned} y &= \frac{1}{10}e^{-2t} + (t-4)e^{-2(t-4)}h(t-4) \\ &= \begin{cases} \frac{1}{10}e^{-2t} & \text{if } 0 \le t < 4 \\ \frac{1}{10}e^{-2t} + (t-4)e^{-2(t-4)} & \text{if } 4 \le t < \infty \end{cases}. \end{aligned}$$

17. Clearly $m = 1$, $\mu = 0$, and $k = 1$. The forcing function is $\delta_0 + \delta_\pi + \cdots + \delta_{5\pi} = \sum_{k=0}^{5} \delta_{\pi k}$. The differential equation that describes the motion is

$$y'' + y = \sum_{k=0}^{5} \delta_{\pi k}.$$

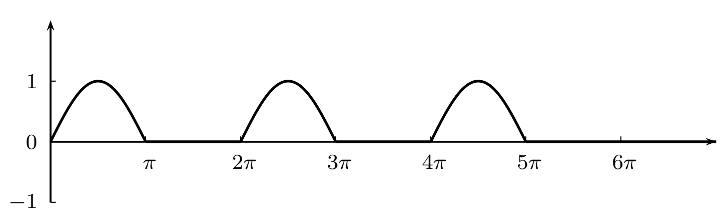
Apply the Laplace transform to get

$$Y(s) = \sum_{k=0}^{5} \frac{e^{-\pi k s}}{s^2+1}.$$

Laplace inversion now gives

$$\begin{aligned} y &= \sum_{k=0}^{5} (\sin(t-\pi k))h(t-\pi k) \\ &= \sum_{k=0}^{5} (-1)^k (\sin t)h(t-\pi k) \\ &= \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } \pi \le t < 2\pi \\ \sin t & \text{if } 2\pi \le t < 3\pi \\ 0 & \text{if } 3\pi \le t < 4\pi \\ \sin t & \text{if } 4\pi \le t < 5\pi \\ 0 & \text{if } 5\pi \le t < \infty \end{cases}. \end{aligned}$$

The graph is given below.



At $t = 0$ the hammer imparts a velocity to the system causing harmonic motion. At $t = \pi$ the hammer strikes in precisely the right way to stop the motion. Then at $t = 2\pi$ the process repeats.

19. The value of $y(t)$ at $t = c$ for y given by Exercise 18 is $y_0 e^{-ac}$. Thus the differential equation we need to solve is $y' + ay = 0$, $y(c) = y_0 e^{-ac} + k$, on the interval $[c, \infty)$. We get the general solution $y(t) = be^{-at}$. The initial condition implies $be^{-ac} = y_0 e^{-ac} + k$. Solving for b gives $b = y_0 + ke^{ac}$. Thus

$$y = y_0 e^{-at} + ke^{-a(t-c)},$$

on the interval $[c, \infty)$.

Section 6.5

1. $F(s) = \frac{1}{s-1}$ and $G(s) = \frac{1}{s} - \frac{e^{-s}}{s}$. Thus $F(s)G(s) = \frac{1}{(s-1)s} - \frac{e^{-s}}{(s-1)s}$. Partial fractions gives $F(s)G(s) = \left(\frac{1}{s-1} - \frac{1}{s}\right) - \left(\frac{1}{s-1} - \frac{1}{s}\right) e^{-s}$ and Laplace inversion gives

$$\begin{aligned} f * g(t) &= e^t - 1 - \left(e^{t-1} - 1\right) h(t-1) \\ &= \begin{cases} e^t - 1 & \text{if } 0 \le t < 1 \\ e^t - e^{t-1} & \text{if } 1 \le t < \infty \end{cases}. \end{aligned}$$

3. $F(s) = e^{-s}\mathcal{L}\{t+1\} = \left(\frac{1}{s^2} + \frac{1}{s}\right) e^{-s}$ and $G(s) = \frac{1}{s}\left(e^{-3s} - e^{-4s}\right)$. Thus

$$F(s)G(s) = \left(\frac{1}{s^3} + \frac{1}{s^2}\right)\left(e^{-4s} - e^{-5s}\right)$$

Laplace inversion now gives

$$\begin{aligned} f * g &= \left(\frac{(t-4)^2}{2} + (t-4)\right) h(t-4) - \left(\frac{(t-5)^2}{2} + (t-5)\right) h(t-5) \\ &= \begin{cases} 0 & \text{if } 0 \le t < 4 \\ \frac{(t-4)^2}{2} + (t-4) & \text{if } 4 \le t < 5 \\ t - 7/2 & \text{if } 5 \le t < \infty \end{cases}. \end{aligned}$$

5. $F(s) = \frac{1}{s} - \frac{1}{s}e^{-2s}$ and $G(s) = \frac{1}{s} - \frac{1}{s}e^{-2s}$. Thus

$$F(s)G(s) = \frac{1}{s^2}\left(1 - 2e^{-2s} + e^{-4s}\right)$$

Laplace inversion now gives

$$\begin{aligned} f * g &= t - 2(t-2)h(t-2) + (t-4)h(t-4) \\ &= \begin{cases} t & \text{if } 0 \le t < 2 \\ -t+4 & \text{if } 2 \le t < 4 \\ 0 & \text{if } 4 \le t < \infty \end{cases}. \end{aligned}$$

7.

$$\begin{aligned} \sin t * (\delta_0 + \delta_\pi) &= \sin t + \sin(t-\pi)h(t-\pi) \\ &= \sin t - (\sin t)h(t-\pi) \\ &= \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } \pi \le t < \infty \end{cases}. \end{aligned}$$

9. The unit impulse response function is $\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$. The homogeneous solution is $y_h = 2e^{3t}$. Observe that $Y_p(s) = \frac{1}{s-3}\frac{e^{-2s}}{s} = \frac{1}{3}\left(\frac{1}{s-3} - \frac{1}{s}\right)e^{-2s}$. It follows that the particular solution is

$$\begin{aligned} y_p &= \zeta * (h(t-2)) \\ &= \mathcal{L}^{-1}\{Y_p(s)\} \\ &= \frac{1}{3}\left(e^{3(t-2)} - 1\right)h(t-2). \end{aligned}$$

and

$$\begin{aligned} y &= y_h + y_p \\ &= 2e^{3t} + \frac{1}{3}\left(e^{3(t-2)} - 1\right)h(t-2) \\ &= \begin{cases} 2e^{3t} & \text{if } 0 \le t < \infty \\ 2e^{3t} + \frac{1}{3}\left(e^{3(t-2)} - 1\right) & \text{if } 1 \le t < \infty \end{cases}. \end{aligned}$$

11. The unit impulse response function is $\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+8}\right\} = e^{-8t}$. The homogeneous solution is $y_h = -2e^{-8t}$. The particular solution is $y_p = \zeta * \chi_{[3,5)}$. Observe that

$$\begin{aligned} Y_p(s) &= \mathcal{L}\{y_p\} \\ &= \frac{1}{s+8}\left(\frac{e^{-3s}}{s} - \frac{e^{-5s}}{s}\right) \\ &= \frac{1}{8}\left(\frac{1}{s} - \frac{1}{s+8}\right)\left(e^{-3s} - e^{-5s}\right). \end{aligned}$$

It follows that

$$\begin{aligned} y_p &= \mathcal{L}^{-1}\{Y_p(s)\} \\ &= \frac{1}{8}\left(1 - e^{-8(t-3)}\right)h(t-3) - \frac{1}{8}\left(1 - e^{-8(t-5)}\right)h(t-5). \end{aligned}$$

and

$$\begin{aligned} y &= y_h + y_p \\ &= -2e^{-8t} + \frac{1}{8}\left(1 - e^{-8(t-3)}\right)h(t-3) - \frac{1}{8}\left(1 - e^{-8(t-5)}\right)h(t-5) \\ &= \begin{cases} -2e^{-8t} & \text{if } 0 \le t < 3 \\ -2e^{-8t} + \frac{1}{8}\left(1 - e^{-8(t-3)}\right) & \text{if } 3 \le t < 5 \\ -2e^{-8t} + \frac{1}{8}\left(e^{-8(t-5)} - e^{-8(t-3)}\right) & \text{if } 5 \le t < \infty \end{cases} \end{aligned}$$

13. The unit impulse response function is $\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin 3t$. The homogeneous solution is $y_h = \cos 3t$. For the particular solution y_p we have

$$\begin{aligned} y_p &= \frac{1}{3}\sin 3t * \chi_{[0,2\pi)} \\ &= \frac{1}{3}\int_0^t \sin(3(t-u))\chi_{[0,2\pi)}\,du \\ &= \frac{1}{3}\begin{cases} \int_0^t \sin(3(t-u))\,du & \text{if } 0 \le t < 2\pi \\ \int_0^{2\pi} \sin(3(t-u))\,du & \text{if } 2\pi \le t < \infty \end{cases} \\ &= \frac{1}{9}\begin{cases} 1 - \cos 3t & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } 2\pi \le t < \infty \end{cases}. \end{aligned}$$

15. Let y be the homogenous solution to $q(\boldsymbol{D})y = 0$ with the given initial conditions. Observe that

$$\mathcal{L}\left\{\boldsymbol{D}^k y\right\} = \begin{cases} s^k Y(s) & \text{if } k < n \\ s^n Y(s) - 1/a_n & \text{if } k = n \end{cases}.$$

and thus

$$\mathcal{L}\left\{a_k \boldsymbol{D}^k y\right\} = \begin{cases} a_k s^k Y(s) & \text{if } \ k < n \\ a_n s^n Y(s) - 1 & \text{if } \ k = n \end{cases}.$$

Therefore $\mathcal{L}\{q(\boldsymbol{D})y\} = q(s)Y(s) - 1 = 0$ from which we get

$$Y(s) = \frac{1}{q(s)}.$$

Hence $y = \zeta$ is the unit impulse response function.

17. Suppose $a_0\zeta + a_1\zeta' + \cdots + a_{n-1}\zeta^{(n-1)} = 0$. Apply the Laplace transform and use Exercise 16 to get

$$\frac{a_0 + a_1 s + \cdots a_{n-1}s^{n-1}}{q(s)} = 0.$$

It follows that the numerator must be identically 0 and hence the coefficients $a_k = 0$, for each k. Thus $\left\{\zeta,\ \zeta', \ldots, \zeta^{(n-1)}\right\}$ is linearly independent.

19. This follows from Exercises 17 and 18.

21. By the input derivative formula we get

$$\mathcal{L}\left\{\boldsymbol{D}^k y\right\} = s^k Y(s) - s^{k-1}y_0 - \cdots - y_{k-1} = s^k Y(s) - \sum_{l=0}^{k-1} s^l y_{k-1-l},$$

for $k \geq 1$. It follows that

$$\mathcal{L}\{q(\boldsymbol{D})y\} = q(s)Y(s) - \sum_{k=1}^{n}\sum_{l=0}^{k-1} a_k s^l y_{k-1-l}.$$

Therefore

$$Y(s) = \sum_{k=1}^{n}\sum_{l=0}^{k-1} a_k \frac{s^l}{q(s)} y_{k-1-l}.$$

Laplace inversion and Exercise 15 give

$$y(t) = \sum_{k=1}^{n}\sum_{l=0}^{k-1} a_k \zeta^{(l)} y_{k-l-1}.$$

Reversing the order of the sum gives

$$\begin{aligned} y(t) &= \sum_{l=0}^{n-1}\left(\sum_{k=l+1}^{n} a_k y_{k-l-1}\right)\zeta^{(l)} \\ &= \sum_{l=0}^{n-1}\left(\sum_{k=0}^{n-l-1} a_{k+l+1} y_k\right)\zeta^{(l)}, \end{aligned}$$

where in the second line and second sum we shifted k to $k+l+1$. It follows that the coefficients are given by

$$c_l = \sum_{k=0}^{n-l-1} a_{k+l+1} y_k.$$

23. We have $q(s) = s^2 - 2s + 1 = (s-1)^2$ and the unit impulse response function is $\zeta = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = te^t$. To compute c_0 we write

$$\begin{array}{ccc} 1 & -2 & 1 \\ & 2 & -3 \end{array}$$

and get $c_0 = -2 \cdot 2 + 1 \cdot (-3) = -7$. For c_1 we consider

$$\begin{array}{cccc} 1 & -2 & 1 & \\ & & 2 & -3 \end{array}$$

and get $c_1 = 1 \cdot 2 = 2$. It follows from Exercise 21 that

$$\begin{aligned} y &= c_0\zeta + c_1\zeta' \\ &= -7te^t + 2(e^t + te^t) \\ &= 2e^t - 5te^t. \end{aligned}$$

25. We have $q(s) = s^3 + s = s(s^2+1)$. Partial fractions give $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$. Thus $\zeta = \mathcal{L}^{-1}\{q(s)\} = 1 - \cos t$. To compute c_0 we write

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ & 1 & 0 & 4 \end{array}$$

and get $c_0 = 1 + 4 = 5$. For c_1 we consider

$$\begin{array}{ccccc} 0 & 1 & 0 & 1 & \\ & & 1 & 0 & 4 \end{array}$$

and get $c_1 = 0$. For c_2 we consider

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ & & & 1 \quad 0 \quad 4 \end{array}$$

and get $c_2 = 1$. It follows from Exercise 21 that

$$\begin{aligned} y &= c_0\zeta + c_1\zeta' + c_2\zeta'' \\ &= 5(1-\cos t) + 0(\sin t) + 1(\cos t) \\ &= 5 - 4\cos t. \end{aligned}$$

Section 6.6

1.

$$\sum_{n=0}^{\infty} (t-n)^2 \chi_{[n,n+1)}(t) \quad = \quad (< t >_1)^2.$$

3.

$$\begin{aligned} \sum_{n=0}^{\infty} n^2 \chi_{[3n,3(n+1))}(t) &= \frac{1}{9}\sum_{n=0}^{\infty} (3n)^2 \chi_{[3n,3(n+1))}(t) \\ &= \frac{1}{9}([t]_3)^2. \end{aligned}$$

5.

$$\begin{aligned} \sum_{n=0}^{\infty} (t+n) \chi_{[2n,2(n+1))}(t) &= \sum_{n=0}^{\infty} (t - 2n + \frac{3}{2}2n) \chi_{[2n,2(n+1))}(t) \\ &= < t >_2 + \frac{3}{2}[t]_2. \end{aligned}$$

7.

$$\begin{aligned} \mathcal{L}\left\{f(< t >_3)\right\} &= \frac{1}{1-e^{-3s}} \mathcal{L}\left\{e^t - e^t h(t-3)\right\} \\ &= \frac{1}{1-e^{-3s}} \left(\frac{1}{s-1} - e^{-3s}\mathcal{L}\left\{e^{t+3}\right\} \right) \\ &= \frac{1}{1-e^{-3s}} \left(\frac{1}{s-1} - \frac{e^{-3s}e^3}{s-1} \right) \\ &= \frac{1-e^{-3(s-1)}}{1-e^{-3s}} \frac{1}{s-1}. \end{aligned}$$

9.

$$\begin{aligned}
\mathcal{L}\{f(<t>_{2p})\} &= \frac{1}{1-e^{-2ps}}\left(\int_0^{2p} e^{-st} f(t)\,dt\right) \\
&= \frac{1}{1-e^{-2ps}}\left(\int_0^{p} e^{-st}\,dt - \int_p^{2p} e^{-st}\,dt\right) \\
&= \frac{1}{1-e^{-2ps}}\left(\frac{e^{-ps}-1}{-s} + \frac{e^{-2ps}-e^{-ps}}{s}\right) \\
&= \frac{(1-e^{-ps})^2}{1-e^{-2ps}}\frac{1}{s} \\
&= \frac{1-e^{-ps}}{1+e^{-ps}}\frac{1}{s}.
\end{aligned}$$

11. Since $<t>_p = t - [t]_p$ we have $[t]_p = t - <t>_p$. Hence

$$\begin{aligned}
\mathcal{L}\{[t]_p\} &= \mathcal{L}\{t\} - \mathcal{L}\{<t>_p\} \\
&= \frac{1}{s^2} - \frac{1}{s^2}\left(1 - \frac{spe^{-ps}}{1-e^{-ps}}\right) \\
&= \frac{pe^{-ps}}{s\,(1-e^{-ps})} \\
&= \frac{p}{s\,(e^{ps}-1)}
\end{aligned}$$

13. On the interval $[2n, 2n+2)$ we have $f(t) = e^{-2n}$ thus

$$f(t) = \sum_{n=0}^{\infty} e^{-2n}\chi_{[2n,2n+2)(t)}.$$

We now have

$$\begin{aligned}
\mathcal{L}\left\{f([t]_2)\right\} &= \sum_{n=0}^{\infty} e^{-2n}\left(\frac{e^{-2ns}-e^{-(2n+2)s}}{s}\right) \\
&= \sum_{n=0}^{\infty} e^{-2n}e^{-2ns}\frac{1-e^{-2s}}{s} \\
&= \frac{1-e^{-2s}}{s}\sum_{n=0}^{\infty} e^{-2n(s+1)} \\
&= \frac{1-e^{-2s}}{s}\sum_{n=0}^{\infty}\left(e^{-2(s+1)}\right)^n \\
&= \frac{1-e^{-2s}}{s}\frac{1}{1-e^{-2(s+1)}} \\
&= \frac{1-e^{-2s}}{1-e^{-2(s+1)}}\frac{1}{s}
\end{aligned}$$

15.

$$\begin{aligned}
\mathcal{L}\{f([t]_p)\} &= \sum_{n=0}^{\infty} f(np)\mathcal{L}\{\chi_{[np,(n+1)p)}\} \\
&= \sum_{n=0}^{\infty} f(np)\frac{e^{-nps}-e^{-(n+1)ps}}{s} \\
&= \frac{1-e^{-ps}}{s}\sum_{n=0}^{\infty} f(np)e^{-nps}.
\end{aligned}$$

17. Let $F(s)=\frac{1-e^{-4(s-2)}}{(1-e^{-4s})(s-2)}$. We first write

$$\begin{aligned}
F(s) &= \sum_{n=0}^{\infty}\frac{1-e^{-4(s-2)}}{s-2}e^{-4ns} \\
&= \sum_{n=0}^{\infty}\frac{e^{-4ns}-e^{8}e^{-4(n+1)s}}{s-2}
\end{aligned}$$

Laplace inversion now gives

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \sum_{n=0}^{\infty} e^{2(t-4n)}h(t-4n) - e^{8}e^{2(t-4(n+1))}h(t-4(n+1)) \\
&= \sum_{n=0}^{\infty} e^{2(t-4n)}\left(h(t-4n) - h(t-4(n+1))\right) \\
&= e^{2t}\sum_{n=0}^{\infty} e^{-2(4n)}\chi_{[4n,4(n+1))} \\
&= e^{2t}e^{-2[t]_4} = e^{2(t-[t]_4)} = e^{2\langle t\rangle_4}.
\end{aligned}$$

19.

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \sum_{n=0}^{\infty}(-1)^n\mathcal{L}^{-1}\left\{\frac{e^{-pns}}{s+a}\right\} \\
&= \sum_{n=0}^{\infty}(-1)^n e^{-a(t-pn)}h(t-pn) \\
&= e^{-at}\sum_{N=0}^{\infty}\sum_{n=0}^{N}(-1)^n e^{apn}\chi_{[N,(N+1)p)} \\
&= e^{-at}\sum_{N=0}^{\infty}\frac{1-(-e^{ap})^{N+1}}{1--e^{ap}} \\
&= e^{-at}\sum_{N=0}^{\infty}\frac{1-(-1)^{N+1}e^{a(N+1)p}}{1+e^{ap}} \\
&= e^{-at}\begin{cases} \frac{1+e^{a(N+1)p}}{1+e^{ap}} & \text{if } t\in[Np,(N+1)p),\ (N \text{ even}) \\ \frac{1-e^{a(N+1)p}}{1+e^{ap}} & \text{if } t\in[Np,(N+1)p),\ (N \text{ odd}) \end{cases} \\
&= e^{-at}\left(\frac{1+(-1)^{\frac{[t]_p}{p}}e^{a([t]_p+p)}}{1+e^{ap}}\right).
\end{aligned}$$

Section 6.7

1. On the interval $[0,2)$ the input rate is $2\cdot 4 = 8$ lbs salt per minute. On the interval $[2,4)$ the input rate is $1\cdot 4 = 4$ lbs salt per minute. The input function $f(t)$ is periodic with period 4. We can thus write $f(t) = 4 + 4\,\mathrm{sw}_2(t)$. The output rate is $\frac{y(t)}{10}\cdot 4$. The resulting differential equation that models this problem is

$$y' + \frac{4}{10}y = 4 + 4\,\mathrm{sw}_2(t), \quad y(0) = 0.$$

Taking the Laplace transform and simplifying gives

$$Y(s) = \frac{4}{s(s+\frac{2}{5})} + \frac{4}{s(s+\frac{2}{5})}\frac{1}{1+e^{-2s}}.$$

Partial fractions gives

$$\frac{4}{s(s+\frac{2}{5})} = \frac{10}{s} - \frac{10}{s+\frac{2}{5}}$$

and hence

$$Y(s) = \frac{10}{s} - \frac{10}{s+\frac{2}{5}} + \frac{10}{s}\frac{1}{1+e^{-2s}} - \frac{10}{s+\frac{2}{5}}\frac{1}{1+e^{-2s}}.$$

Let

$$\begin{aligned} Y_1(s) &= \frac{10}{s} - \frac{10}{s+\frac{2}{5}} \\ Y_2(s) &= \frac{10}{s}\frac{1}{1+e^{-2s}} \\ Y_3(s) &= \frac{10}{s+\frac{2}{5}}\frac{1}{1+e^{-2s}}. \end{aligned}$$

Example 6.6.2 and Exercise 6.6.19 are useful for taking the inverse Laplace transforms of Y_2 and Y_3. We get

$$\begin{aligned} y_1(t) &= 10 - 10e^{\frac{-2t}{5}} \\ y_2(t) &= 10\,\mathrm{sw}_2(t) \\ y_3(t) &= \frac{10e^{\frac{-2t}{5}}}{1+e^{\frac{4}{5}}}\left(1 + e^{\frac{4}{5}}\begin{cases} e^{\frac{4N}{5}} & \text{if } t \in [2N, 2(N+1)), \text{ N even} \\ -e^{\frac{4N}{5}} & \text{if } t \in [2N, 2(N+1)), \text{ N odd} \end{cases}\right) \\ &= \frac{10e^{\frac{-2t}{5}}}{1+e^{\frac{4}{5}}}\left(1 + e^{\frac{4}{5}}(-1)^{[t/2]_1}e^{\frac{2}{5}[t]_2}\right). \end{aligned}$$

It now follows that

$$\begin{aligned} y(t) &= y_1(t) + y_2(t) - y_3(t) \\ &= 10 - 10e^{\frac{-2t}{5}} + 10\,\mathrm{sw}_2(t) - \frac{10e^{\frac{-2t}{5}}}{1+e^{\frac{4}{5}}}\left(1 + e^{\frac{4}{5}}(-1)^{[t/2]_1}e^{\frac{2}{5}[t]_2}\right) \end{aligned}$$

When $t = 2N$ and N is even then

$$y(2N) = 20 - 10e^{\frac{-2}{5}2N} - 10\frac{e^{\frac{-2}{5}2N}}{1+e^{\frac{4}{5}}}\left(1 + e^{\frac{4}{5}}e^{\frac{2}{5}2N}\right).$$

Continuing $y(2N)$ to all $t \geq 0$ gives

$$l(t) = 20 - 10e^{\frac{-2}{5}t} - 10\frac{e^{\frac{-2}{5}t}}{1+e^{\frac{4}{5}}}\left(1 + e^{\frac{4}{5}}e^{\frac{2}{5}t}\right),$$

a function whose graph bounds the graph of y from below. In a similar way for $t = 2N$, N odd, we get

$$u(t) = 10 - 10e^{\frac{-2}{5}t} - 10\frac{e^{\frac{-2}{5}t}}{1+e^{\frac{4}{5}}}\left(1 - e^{\frac{4}{5}}e^{\frac{2}{5}t}\right),$$

whose graph bounds the graph of y from above. Now observe that

$$\lim_{t\to\infty} l(t) = 20 - \frac{10e^{\frac{4}{5}}}{1+e^{\frac{4}{5}}} \approx 13.10 \quad \text{and} \quad \lim_{t\to\infty} u(t) = 10 + \frac{10e^{\frac{4}{5}}}{1+e^{\frac{4}{5}}} \approx 16.9$$

Thus the amount of salt fluctuates from 13.10 pounds to 16.90 pounds in the long term.

3. The input function is $5\sum_{n=1}^{\infty} \delta_{2n} = 5\delta_0(< t >_2)$. and therefore the differential equation that models this system is

$$y' + \frac{1}{2}y = 5\delta_0(< t >_2), \quad y(0) = 0.$$

By Proposition 6.6.6 the Laplace transform gives

$$Y(s) = \frac{5}{s+\frac{1}{2}}\frac{1}{1-e^{-2s}}.$$

By Theorem 6.6.7 Laplace inversion gives

$$\begin{aligned} y(t) &= 5\sum_{N=0}^{\infty}\left(\sum_{n=0}^{N} e^{-\frac{1}{2}(t-2n)}\right)\chi_{[2N,2(N+1))} \\ &= 5e^{-\frac{1}{2}t}\sum_{N=0}^{\infty}\frac{e^{N+1}-1}{e-1}\chi_{[2N,2(N+1))} \\ &= 5e^{-\frac{1}{2}t}\frac{e^{\frac{1}{2}[t]_2+1}-1}{e-1}. \end{aligned}$$

The solution is sandwiched in between a lower and upper curve. The upper curve, $u(t)$, is obtained by setting $t = 2m$ to be an even integer in the formula for the solution and then continuing it to all reals. We obtain

$$u(2m) = 5e^{-\frac{1}{2}2m}\frac{e^{\frac{1}{2}[2m]_2+1}-1}{e-1} = 5e^{-\frac{1}{2}2m}\frac{e^{\frac{1}{2}2m+1}-1}{e-1}.$$

Thus

$$u(t) = 5e^{-\frac{1}{2}t}\frac{e^{\frac{1}{2}t+1}-1}{e-1} = 5\frac{e-e^{-\frac{1}{2}t}}{e-1}.$$

In a similar way, the lower curve, $l(t)$, is obtained by setting $t = 2(m+1)^-$ (slightly less than the even integer $2(m+1)$) and continuing to all reals. We obtain

$$l(t) = 5\frac{1-e^{-\frac{1}{2}t}}{e-1}.$$

An easy calculation gives

$$\lim_{t\to\infty} u(t) = 5\tfrac{e}{e-1} \simeq 7.91 \quad \text{and } \lim_{t\to\infty} l(t) = 5\tfrac{1}{e-1} \simeq 2.91.$$

This means that the salt fluctuation in the tank varies between 2.91 and 7.91 pounds for large values of t.

5. Let $y(t)$ be the number of allegators at time t measured in months. We assume the Malthusian growth model $y' = ry$. Thus $y(t) = y(0)e^{rt} = 3000e^{rt}$. To determine the growth rate r we know $y(-12) = 2500$ (12 months earlier there were 2500 allegators). Thus $2500 = 3000e^{-12r}$ and hence $r = \frac{1}{12}\ln\frac{6}{5}$. The elite force of Cajun allegator hunters instantaneously remove 40 allegators at the beginning of each month. This can be modeled by $40(\delta_0+\delta_1+\cdots) = 40\delta_0(< t >_1)$. The mathematical model is thus

$$y' = ry - 40\delta_0(< t >_1), \quad y(0) = 3000,$$

where $r = \frac{1}{12}\ln\frac{6}{5}$. We apply the Laplace transform and use Proposition 6.6.6 to get

$$Y(s) = \frac{3000}{s-r} - 40\frac{1}{s-r}\frac{1}{1-e^{-s}}.$$

Let

$$\begin{aligned} Y_1(s) &= \frac{3000}{s-r} \\ Y_2(s) &= 40\frac{1}{s-r}\frac{1}{1-e^{-s}} \end{aligned}$$

Then $Y(s) = Y_1(s) + Y_2(s)$. We use Theorem 6.6.7 to get

$$\begin{aligned} y_1(t) &= 3000e^{rt} \\ y_2(t) &= 40\sum_{N=0}^{\infty}\left(\sum_{n=0}^{N} e^{r(t-n)}\right)\chi_{[N,N+1)} \\ &= 40e^{rt}\sum_{N=0}^{\infty}\frac{1-e^{-r(N+1)}}{1-e^{-r}}\chi_{[N,N+1)} \\ &= 40\frac{e^{rt}-e^{-r([t]_1-t+1)}}{1-e^{-r}}. \end{aligned}$$

It follows now that

$$\begin{aligned} y(t) &= y_1(t)-y_2(t) \\ &= 3000e^{rt}-40\frac{e^{rt}-e^{-r([t]_1-t+1)}}{1-e^{-r}}. \end{aligned}$$

To determine the population at the beginning of 5 years = 60 months we compute

$$\begin{aligned} y(60) &= 3000e^{60r}-40\frac{e^{60r}-e^{-r}}{1-e^{-r}} \\ &= 7464.96-3988.16 \\ &\approx 3477 \end{aligned}$$

Section 6.8

1. Since $c\beta = \sqrt{2}$ is not an odd multiple of π we get

$$\begin{aligned} y(t) &= 2\left(2\,\mathrm{sw}_1(t)-(-1)^{[t]_1}(\cos < t >_1 - \alpha\sin < t >_1)\right) \\ &\quad -2\left(\cos t+\alpha\sin t\right), \end{aligned}$$

where $\alpha = \frac{-\sin\sqrt{2}}{1+\cos\sqrt{2}}$. Since $\frac{\beta c}{\pi} = \frac{2}{\pi}$ is irrational the motion is non periodic

3. Since $c\beta = 2\pi$ is not an odd multiple of π we get

$$\begin{aligned} y(t) &= \frac{1}{\pi^2}\left(2\,\mathrm{sw}_2(t)-(-1)^{[t/2]_1}\cos\pi < t >_2 - \cos\pi t\right) \\ &= \frac{1}{\pi^2}\left(2\,\mathrm{sw}_2(t)-\cos\pi t\left((-1)^{[t/2]_1}+1\right)\right), \end{aligned}$$

where we have used the identity $\cos\pi < t >_2 = \cos\pi t$. Since $\frac{\beta c}{\pi} = 2$ is rational the motion is periodic.

5. Since $c\beta = \pi$ is an odd multiple of π we get $y(t) = \frac{2}{\pi^2}\left(\mathrm{sw}_1(t) - [t]_1 \cos \pi t - \cos \pi t\right)$. Resonance occurs.

7. Since $c\beta = \pi$ is not a multiple of 2π and $\gamma = 0$ we get

$$\begin{aligned} y(t) &= \sin t + \sin < t >_\pi \\ &+ (1 + (-1)^{[t/\pi]_1}) \sin t, \end{aligned}$$

where we have used $\sin < t >_\pi = (-1)^{[t/\pi]_1} \sin t$. Since $\frac{\beta c}{2\pi} = \frac{1}{2}$ is rational the motion is periodic.

9. Since $c\beta = 1$ is not a multiple of 2π and $\gamma = \frac{-\sin 1}{1-\cos 1}$ we get

$$y(t) = \sin t + \gamma \cos t + \sin < t >_1 - \gamma \cos < t >_1,$$

where $\gamma = \frac{-\sin 1}{1-\cos 1}$. Since $\frac{\beta c}{2\pi} = \frac{1}{2\pi}$ is not rational the motion is non periodic.

11. Since $\beta c = 2\pi$ we get

$$y(t) = 2(\sin t)(1 + [t/2\pi]_1).$$

Resonance occurs.

13. First we have

$$\begin{aligned} \sum_{n=0}^{N} \left(e^{i\theta}\right)^n &= \sum_{n=0}^{N} e^{in\theta} \\ &= \sum_{n=0}^{N} \cos n\theta + i \sin n\theta \\ &= \sum_{n=0}^{N} \cos n\theta + i \sum_{n=0}^{N} \sin n\theta. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{n=0}^{N} \left(e^{i\theta}\right)^n &= \frac{e^{i(N+1)\theta} - 1}{e^{i\theta} - 1} \\ &= \frac{\cos(N+1)\theta - 1 + i\sin(N+1)\theta}{\cos\theta - 1 + i\sin\theta} \\ &= \frac{\cos(N+1)\theta - 1 + i\sin(N+1)\theta}{\cos\theta - 1 + i\sin\theta} \cdot \frac{\cos\theta - 1 - i\sin\theta}{\cos\theta - 1 - i\sin\theta} \end{aligned}$$

The product of the denominators simplifies to $2 - 2\cos\theta$. The product of the numerators has a real and imaginary part. Call them R and I, respectively. Then

$$\begin{aligned} R &= (\cos((N+1)\theta)-1)(\cos\theta-1)+\sin((N+1)\theta)\sin\theta \\ &= \cos((N+1)\theta)\cos\theta+\sin((N+1)\theta)\sin\theta-\cos(N+1)\theta-\cos\theta+1 \\ &= \cos(N\theta)-\cos(N\theta)\cos\theta+\sin(N\theta)\sin\theta-\cos\theta+1 \\ &= (\cos(N\theta)+1)(1-\cos\theta)+\sin(N\theta)\sin\theta \end{aligned}$$

and

$$\begin{aligned} I &= \sin((N+1)\theta)(\cos\theta-1)-\sin\theta(\cos((N+1)\theta)-1) \\ &= \sin((N+1)\theta)\cos\theta-\sin((N+1)\theta)+\sin\theta-\cos((N+1)\theta)\sin\theta \\ &= \sin(N\theta)-\sin(N\theta)\cos\theta-\cos(N\theta)\sin\theta+\sin\theta \\ &= \sin(N\theta)(1-\cos\theta)+\sin\theta(1-\cos(N\theta)). \end{aligned}$$

Equating real and imaginary parts and simplifying now gives

$$\begin{aligned} \sum_{n=0}^{N}\cos n\theta &= \frac{R}{2-2\cos\theta} \\ &= \frac{(\cos(N\theta)+1)(1-\cos\theta)+\sin(N\theta)\sin\theta}{2-2\cos\theta} \\ &= \frac{1}{2}\left(1+\cos N\theta+\gamma\sin N\theta\right) \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{N}\sin n\theta &= \frac{I}{2-2\cos\theta} \\ &= \frac{\sin(N\theta)(1-\cos\theta)+\sin\theta(1-\cos(N\theta))}{2-2\cos\theta} \\ &= \frac{1}{2}\left(\sin N\theta+\gamma(1-\cos N\theta)\right) \end{aligned}$$

15. Let

$$\begin{aligned} R(v) &= \sum_{n=0}^{N}\cos nv = \frac{1}{2}\left(1+\cos Nv+\gamma\sin Nv\right) = \mathrm{Re}\sum_{n=0}^{N}e^{inv} \\ I(v) &= \sum_{n=0}^{N}\sin nv = \frac{1}{2}\left(\sin Nv+\gamma(1-\cos Nv)\right) = \mathrm{Im}\sum_{n=0}^{N}e^{inv}, \end{aligned}$$

as in Exercise 13. Now

$$\begin{aligned}
\sum_{n=0}^{N} \cos(u+nv) &= \operatorname{Re} \sum_{n=0}^{N} e^{i(u+nv)} \\
&= \operatorname{Re}\left(e^{iu} \sum_{n=0}^{N} e^{inv} \right) \\
&= \operatorname{Re}\left((\cos u + i \sin u)(R(v) + iI(v)) \right) \\
&= (\cos u) R(v) - (\sin u) I(v) \\
&= \frac{1}{2}\left(\cos u + \cos u \cos Nv + \gamma \cos u \sin Nv\right) \\
&\quad -\frac{1}{2}\left(\sin u \sin Nv + \gamma(\sin u - \sin u \cos Nv)\right) \\
&= \frac{1}{2}\left(\cos u + \cos(u+Nv) + \gamma(-\sin u + \sin(u+Nv)\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{n=0}^{N} \sin(u+nv) &= \operatorname{Im} \sum_{n=0}^{N} e^{i(u+nv)} \\
&= \operatorname{Im}\left(e^{iu} \sum_{n=0}^{N} e^{inv} \right) \\
&= \operatorname{Im}\left((\cos u + i \sin u)(R(v) + iI(v)) \right) \\
&= (\sin u) R(v) + (\cos u) I(v) \\
&= \frac{1}{2}\left(\sin u + \sin u \cos Nv + \gamma \sin u \sin Nv\right) \\
&\quad +\frac{1}{2}\left(\cos u \sin Nv + \gamma(\cos u - \cos u \cos Nv)\right) \\
&= \frac{1}{2}\left(\sin u + \sin(u+Nv) + \gamma(\cos u - \cos(u+Nv))\right).
\end{aligned}$$

Section 7.1

1. The ratio test gives $\frac{(n+1)^2}{n^2} \to 1$. $R = 1$.

3. The ratio test gives $\frac{2^n n!}{2^{n+1}(n+1)!} = \frac{1}{2(n+1)} \to 0$. $R = \infty$.

5. The ratio test gives $\frac{(n+1)!}{n!} = n + 1 \to \infty$. $R = 0$.

7. t is a factor in this series which we factor out to get $t \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$. Since t is a polynomial its presence will not change the radius of convergence.

Let $u = t^2$ in the new powers series to get $\sum_{n=0}^{\infty} \frac{(-1)^n u^n}{(2n)!}$. The ratio test gives $\left|\frac{(-1)^{n+1}(2n)!}{(-1)^n(2n+2)!}\right| = \frac{1}{(2n+1)(2n+2)} \to 0$. The radius of convergence in u and hence t is ∞.

9. The expression in the denominator can be written $\frac{1\cdot 3\cdot 5\cdots(2n+1)}{1} = \frac{1\cdot 2\cdot 3\cdot 4\cdots(2n+1)}{2\cdot 4\cdots 2n} = \frac{(2n+1)!}{2^n(1\cdot 2\cdot 3\cdots n)} = \frac{(2n+1)!}{2^n n!}$ and the given power series is $\sum_{n=0}^{\infty} \frac{(n!)^2 2^n t^n}{(2n+1)!}$. The ratio test gives $\frac{((n+1)!)^2 2^{n+1}(2n+1)!}{(n!)^2 2^n (2n+3)!} = \frac{(n+1)^2 2}{(2n+3)(2n+2)} \to \frac{1}{2}$. $R = 2$.

11. Use the geometric series to get $\frac{1}{t-a} = \frac{-1}{a}\frac{1}{1-\frac{t}{a}} = \frac{-1}{a}\sum_{n=0}^{\infty}\left(\frac{t}{a}\right)^n = -\sum_{n=0}^{\infty}\frac{t^n}{a^{n+1}}$.

13. $\frac{\sin t}{t} = \frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n+1)!}$.

15. Recall that $\tan^{-1} t = \int \frac{1}{1+t^2}\,dt$. Using the result of Exercise 10 we get $\tan^{-1} t = \sum_{n=0}^{\infty}\int(-1)^n t^{2n}\,dt + C = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{2n+1} + C$. Since $\tan^{-1} 0 = 0$ it follows that $C = 0$. Thus $\tan^{-1} t = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{2n+1}$.

17. Since $\tan t$ is odd we can write $\tan t = \sum_{n=0}^{\infty} d_{2n+1}t^{2n+1}$ and hence $\sin t = \cos t\sum_{n=0}^{\infty} d_{2n+1}t^{2n+1}$. Writing out a few terms gives $t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots = (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots)(d_1 t + d_3 t^3 + d^5 t^5 \cdots)$. Collecting like powers of t gives the following recursion relations

$$\begin{aligned} d_1 &= 1 \\ d_3 - \frac{d_1}{2!} &= \frac{-1}{3!} \\ d_5 - \frac{d_3}{2!} + \frac{d_1}{4!} &= \frac{1}{5!} \\ d_7 - \frac{d_5}{2!} + \frac{d_3}{4!} - \frac{d_1}{6!} &= \frac{-1}{7!}. \end{aligned}$$

Solving these equations gives

$$\begin{aligned} d_1 &= 1 \\ d_3 &= \frac{1}{3} \\ d_5 &= \frac{2}{15} \\ d_7 &= \frac{17}{315}. \end{aligned}$$

Thus $\tan t = 1 + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \cdots$.

19. $e^t \sin t = (1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots)(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots) = t + (1)t^2 + (\frac{-1}{3!} + \frac{1}{2!})t^3 + (\frac{-1}{3!} + \frac{1}{3!})t^4 + (\frac{1}{5!} - \frac{1}{2!}\frac{1}{3!} + \frac{1}{4!})t^5 \cdots = t + t^2 + \frac{1}{3}t^3 - \frac{1}{30}t^5$.

21. The ratio test gives infinite radius of convergence. Let $f(t)$ be the function defined by the given power series. Then

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty}(-1)^n \frac{n+1}{n!} t^n \\ &= \sum_{n=0}^{\infty}(-1)^n \left(\frac{n}{n!} + \frac{1}{n!}\right) t^n \\ &= \sum_{n=1}^{\infty}(-1)^n \frac{1}{(n-1)!} t^n + \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!} t^n \\ &= t\sum_{n=0}^{\infty}(-1)^{n+1} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \\ &= -te^{-t} + e^{-t} \end{aligned}$$

23. It is easy to check that the interval of convergence is $(-1, 1)$. Let $f(t)$ be the function defined by the given power series. Then

$$\begin{aligned} \int f(t)\, dt &= \sum_{n=0}^{\infty}(n+1)\frac{t^{n+1}}{n+1} + c \\ &= t\sum_{n=0}^{\infty} t^n + c \\ &= \frac{t}{1-t} + c. \end{aligned}$$

Differentiation gives

$$f(t) = \frac{1}{(1-t)^2}.$$

25. It is not hard to see that the interval of convergence is $(-1,1)$. Let $f(t)$ be the given power series. A partial fraction decomposition gives

$$\frac{1}{(2n+1)(2n-1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

Therefore

$$f(t) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{2n-1} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{2n+1}.$$

Let $f_1(t) = \sum_{n=0}^{\infty}\frac{t^{2n+1}}{2n-1}$ and $f_2(t) = \sum_{n=0}^{\infty}\frac{t^{2n+1}}{2n+1}$. Then $f_1(t) = -t + \frac{t^2}{2}\ln\frac{1+t}{1-t}$ by Exercise 24. Observe that $f_2(0) = 0$ and

$$\begin{aligned} f_2'(t) &= \sum_{n=0}^{\infty} t^{2n} \\ &= \frac{1}{1-t^2} = \frac{1}{2}\frac{1}{1+t} + \frac{1}{2}\frac{1}{1-t}. \end{aligned}$$

Integration and the fact that $f_2(0) = 0$ gives $f_2(t) = \frac{1}{2}\ln\frac{1+t}{1-t}$. It follows now that

$$\begin{aligned} f(t) &= \frac{1}{2}(f_1(t) - f_2(t)) \\ &= \frac{1}{2}\left(-t + \frac{t^2}{2}\ln\frac{1+t}{1-t} - \frac{1}{2}\ln\frac{1+t}{1-t}\right) \\ &= \frac{-t}{2} + \frac{t^2-1}{4}\ln\frac{1+t}{1-t}. \end{aligned}$$

27. The binomial theorem: $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$.

29. The ratio $\frac{c_{n+1}}{c_n}$ is $\frac{1}{2}$ if n is even and 2 if n is odd. Thus $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ does not exist. The ratio test does not apply. The root test gives that $\sqrt[n]{c_n}$ is 1 if n is odd and $\sqrt[n]{2}$ if n is even. As n approaches ∞ both even and odd terms approach 1. It follows that the radius of convergence is 1.

31. Suppose $f^{(n)}(t) = e^{\frac{-1}{t}}p_n(\frac{1}{t})$ where p_n is a polynomial. Then $f^{(n+1)}(t) = e^{\frac{-1}{t}}\left(\frac{1}{t^2}\right)p_n(\frac{1}{t}) + p_n'(\frac{1}{t})\left(\frac{-1}{t^2}\right)e^{\frac{-1}{t}} = e^{\frac{-1}{t}}p_{n+1}(\frac{1}{t})$, where $p_{n+1}(x) = x^2(p_n(x) - p_n'(x))$. By mathematical induction it follows that $f^{(n)}(t) = e^{\frac{-1}{t}}p_n\left(\frac{1}{t}\right)$ for all $n = 1, 2, \ldots$.

33. Since $f(t) = 0$ for $t \le 0$ clearly $\lim_{t\to 0^-} f^{(n)}(t) = 0$ The previous problems imply that the right hand limits are also zero. Thus $f^{(n)}(0)$ exist and is 0.

Section 7.2

1. Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$. Then $c_{n+2} = \frac{c_n}{(n+2)(n+1)}$. Consider even and odd cases to get $c_{2n} = \frac{c_0}{(2n)!}$ and $c_{2n+1} = \frac{c_1}{(2n+1)!}$. Thus $y(t) = c_0 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = c_0 \cosh t + c_1 \sinh t$. (see Example 7.1.7) We observe that the characteristic polynomial is $s^2 - 1 = (s-1)(s+1)$ so $\{e^t, e^{-t}\}$ is a fundamental set. But $\cosh t = \frac{e^t + e^{-t}}{2}$ and $\sinh t = \frac{e^t - e^{-t}}{2}$; the set $\{\cosh t, \sinh t\}$ is also a fundamental set.

3. Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$. Then $c_{n+2}(n+2)(n+1) + k^2 c_n = 0$ or $c_{n+2} = -\frac{k^2 c_n}{(n+2)(n+1)}$. We consider first the even case.

$$\begin{array}{ll} n = 0 & c_2 = -\frac{k^2 c_0}{(2\cdot 1} \\ n = 2 & c_4 = -\frac{k^2 c_2}{4\cdot 3} = \frac{k^4 c_0}{4!} \\ n = 4 & c_6 = -\frac{k^6 c_0}{6!} \\ \vdots & \vdots \end{array}$$

From this it follows that $c_{2n} = (-1)^n \frac{k^{2n} c_0}{(2n)!}$. The odd case is similar. We get $c_{2n+1} = (-1)^n \frac{k^{2n+1}}{(2n+1)!}$. The power series expansion becomes

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} c_n t^n \\ &= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n} t^{2n}}{(2n)!} \\ &+ c_1 \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n+1} t^{2n+1}}{(2n+1)!} \\ &= c_0 \cos kt + c_1 \sin kt. \end{aligned}$$

5. Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$. Then the recurrence relation is

$$(n+2)(n+1)c_{n+2} - (n-2)(n+1)c_n = 0$$

or

$$c_{n+2} = \frac{n-2}{n+2} c_n.$$

Since there is a difference of two in the indices we consider the even and odd case. We consider first the even case.

$$\begin{array}{ll} n=0 & c_2 = -c_0 \\ n=2 & c_4 = \frac{0}{4}c_2 = 0 \\ n=4 & c_6 = \frac{2}{6}c_4 = 0 \\ \vdots & \vdots \end{array}$$

It follows that $c_{2n} = 0$ for all $n = 2, 3, \ldots$. Thus

$$\begin{aligned} \sum_{n=0}^{\infty} c_{2n}t^{2n} &= c_0 + c_2t^2 + 0t^4 + \cdots \\ &= c_0(1-t^2) \end{aligned}$$

and hence $y_0(t) = 1 - t^2$. We now consider the odd case.

$$\begin{array}{ll} n=1 & c_3 = \frac{-1}{3}c_1 \\ n=3 & c_5 = \frac{1}{5}c_3 = -\frac{1}{5\cdot 3}c_1 \\ n=5 & c_7 = \frac{3}{7}c_5 = -\frac{1}{7\cdot 5}c_1 \\ n=7 & c_9 = \frac{5}{9}c_5 = -\frac{1}{9\cdot 7}c_1 \\ \vdots & \vdots \end{array}$$

From this we see that $c_{2n+1} = \frac{-c_1}{(2n+1)(2n-1)}$. Thus

$$\sum_{n=0}^{\infty} c_{2n+1}t^{2n+1} = -c_1 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)(2n-1)}$$

and hence $y_1(t) = -\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)(2n-1)}$. By Exercise 7.1.25 we can write y_1 as

$$y_1(t) = \frac{t}{2} - \frac{t^2-1}{4}\ln\left(\frac{1+t}{1-t}\right).$$

The general solution is

$$y(t) = c_0(1-t^2) - c_1\left(\frac{t}{2} + \frac{t^2-1}{4}\ln\left(\frac{1-t}{1+t}\right)\right).$$

(See also Exercise 5.5.15.)

7. Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$. Then the recurrence relation is

$$c_{n+2} = \frac{2}{n+2}c_{n+1} - \frac{n-1}{(n+2)(n+1)}c_n.$$

For the first several terms we get

$$\begin{array}{ll} n=0 & c_2 = 0c_1 + \frac{1}{2}c_0 = \frac{1}{2!}c_0 \\ n=1 & c_3 = \frac{1}{2}c_2 - 0 = \frac{1}{3!}c_0 \\ n=2 & c_4 = \frac{2}{4}c_3 - \frac{1}{4\cdot 3}c_2 = \frac{1}{4!}c_0 \\ n=3 & c_5 = \frac{3}{5}c_4 - \frac{2}{5\cdot 4}c_3 = \frac{3}{5!}c_0 - \frac{2}{5!}c_0 = \frac{1}{5!}c_0 \\ \vdots & \vdots \end{array}$$

In general,

$$c_n = \frac{1}{n!}c_0, \quad n = 2, 3, \ldots.$$

We now get

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} c_n t^n \\ &= c_0 + c_1 t + \sum_{n=2}^{\infty} c_n t^n \\ &= (c_1 - c_0)t + c_0 + c_0 t + c_0 \sum_{n=2}^{\infty} \frac{t^n}{n!} \\ &= (c_1 - c_0)t + c_0 e^t \\ &= c_0(e^t - t) + c_1 t. \end{aligned}$$

9. Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$. Then the recurrence relation is

$$c_{n+2} = -\frac{(n-2)(n-3)}{(n+2)(n+1)}c_n.$$

The even case gives:

$$\begin{array}{ll} n=0 & c_2 = -\frac{6}{2}c_0 = -3c_0 \\ n=2 & c_4 = 0 \\ n=4 & c_6 = 0 \\ \vdots & \vdots \end{array}$$

Hence

$$\sum_{n=0}^{\infty} c_{2n} t^{2n} = c_0 + c_2 t^2 = c_0(1 - 3t^2).$$

The odd case gives

$$\begin{array}{ll} n=1 & c_3 = -\frac{1}{3}c_1 \\ n=3 & c_5 = 0 \\ n=5 & c_7 = 0 \\ \vdots & \vdots \end{array}$$

Hence

$$\sum_{n=0}^{\infty} c_{2n+1}t^{2n+1} = c_1 t + c_3 t^3 = c_1(t - \frac{t^3}{3}).$$

The general solution is

$$y(t) = c_0(1-3t^2) + c_1(t - \frac{t^3}{3}).$$

11. Let n be an integer. Then $e^{inx} = (e^{ix})^n$. By Euler's formula this is

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx.$$

13. By de Moivre's formula $\sin(n+1)x$ is the imaginary part of $(\cos x + i \sin x)^{n+1}$. The binomial theorem gives

$$\begin{aligned} \cos(n+1)x + i\sin(n+1)x &= (\cos x + i \sin x)^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} \cos^{n+1-k} x i^k \sin^k x \end{aligned}$$

Only the odd powers of i contribute to the imaginary part. It follows that

$$\begin{aligned} \sin(n+1)x &= \operatorname{Im} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} \cos^{n+1-(2j+1)} x (i^{2j+1}) \sin^{2j+1} x \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n+1}{2j+1} \cos^{n-2j} x (1-\cos^2 x)^j \sin x, \end{aligned}$$

where we use the greatest integer function $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x. Now replace $t = \cos x$ and using the definition $\sin x U_n(cosx) = \sin(n+1)x$ to get $U_n(t) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n+1}{2j+1} t^{n-2j}(1-t^2)^j$. It follows that $\sin nx$ is a product of $\sin x$ and a polynomial in $\cos x$.

15. We have

$$\sin((n+2)x) = \sin((n+1)x+x) = \sin((n+1)x)\cos x + \cos((n+1)x)\sin x$$

and hence

$$(\sin x)U_{n+1}(\cos x) = (\sin x)U_n(x)\cos x + (\sin x)T_{n+1}(\cos x).$$

Now divide by $\sin x$ and let $t = \cos x$. We get

$$U_{n+1}(t) = tU_n(t) + T_{n+1}(t).$$

17. By using the sum and difference formula it is easy to verify the following trigonometric identity:

$$2\sin a \cos b = \sin(a+b) + \sin(a-b).$$

Let $a = (n+1)x$ and $b = x$. Then

$$2\cos x \sin(n+1)x = \sin((n+2)x) + \sin(nx)$$

and hence

$$2(\cos x)U_n(\cos x)/\sin x = U_{n+1}(\cos x)/\sin x + U_{n-1}(\cos x)/\sin x.$$

Now cancel out $\sin x$ and let $t = \cos x$.

19. By using the sum and difference formula it is easy to verify the following trigonometric identity:

$$2\sin a \sin b = \cos(b-a) - \cos(a+b).$$

Let $a = x$ and $b = nx$. Then

$$2\sin x \sin nx = \cos((n-1)x) - \cos((n+1)x)$$

and hence

$$2U_{n-1}(\cos x) = T_{n-1}(\cos x) - T_{n+1}(\cos x).$$

Now let $t = \cos x$, replace n by $n+1$, and divide by 2.

Section 7.3

1. The function $\frac{t}{1-t^2}$ is analytic except at $t = 1$ and $t = -1$. The function $\frac{1}{1+t}$ is analytic except at $t = -1$. It follows that $t = 1$ and $t = -1$ are the only singular points. Observe that $(t-1)\left(\frac{t}{1-t^2}\right) = \frac{-t}{1+t}$ is analytic at 1 and $(t-1)^2\left(\frac{1}{1+t}\right)$ is analytic at $t=1$. It follows that 1 is a regular singular point. Also observe that $(t+1)\left(\frac{t}{1-t^2}\right)=\frac{t}{1-t}$ is analytic at -1 and $(t+1)^2\left(\frac{1}{1+t}\right)=(1+t)$ is analytic at $t=-1$. It follows that -1 is a regular singular point. Thus 1 and -1 are regular points.

3. Both $3t(1-t)$ and $\frac{1-e^t}{t}$ are analytic. There are no singular points and hence no regular points.

5. We first write it in standard form: $y''+\frac{1-t}{t}y'+4y=0$. While the coefficient of y is analytic the coefficient of y' is $\frac{1-t}{t}$ is analytic except at $t=0$. It follows that $t=0$ is a singular point. Observe that $t\left(\frac{1-t}{t}\right)=1-t$ is analytic and $4t^2$ is too. It follows that $t=0$ is a regular point.

7. The indicial equation is $q(s)=s(s-1)+2s=s^2+s=s(s+1)$ The exponents of singularity are 0 and -1. Theorem 2 guarantees one Frobenius solution but there could be two.

9. In standard form the equation is $t^2y''+t(1-t)y'+\lambda ty=0$. The indicial equation is $q(s)=s(s-1)+s=s^2$ The exponent of singularity is 0 with multiplicity 2. Theorem 2 guarantees that there is one Frobenius solution. The other has a logarithmic term.

11. $y_1(t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n+1)!}=\frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}=\frac{1}{t}\sin t$. y_2 is done similarly.

13. Let $y(t)=t^{-2}v(t)$. Then $y'(t)=-2t^{-3}v(t)+t^{-2}v'(t)$ and $y''(t)=6t^{-4}v(t)-4t^{-3}v'(t)+t^{-2}v''(t)$. From which we get

$$\begin{aligned}t^2y'' &= 6t^{-2}v(t)-4t^{-1}v'(t)+v''(t)\\ 5ty' &= -10t^{-2}v(t)+5t^{-1}v'(t)\\ 4y &= 4t^{-2}v(t).\end{aligned}$$

Adding these terms and remembering that we are assuming the y is a solution we get

$$0=t^{-1}v'(t)+v''(t).$$

From this we get $\frac{v''}{v'}=\frac{-1}{t}$. Integrating we get $\ln v'(t)=-\ln t$ and hence $v'(t)=\frac{1}{t}$. Integrating again gives $v(t)=\ln t$. It follows that $y(t)=t^{-2}\ln t$ is a second independent solution. The indicial polynomial is $q(s)=s(s-1)+5s+4=(s-2)^2$. Case 3 of the theorem guarantees that one solution is a Frobenius solution and the other has logarithmic term.

In each case below we let $y=t^r\sum_{n=0}^{\infty}c_nt^n$ where we assume $c_0\neq 0$ and r is the exponent of singularity.

15. Indicial polynomial: $p(s) = s(s-3)$; exponents of singularity $s = 0$ and $s = 3$.

$$\begin{array}{ll} n=0 & c_0(r)(r-3)=0 \\ n=1 & c_1(r-2)(r+1)=0 \\ n\geq 1 & c_n(n+r)(n+r-3)=-c_{n-1} \end{array}$$

r=3:

$$\begin{array}{ll} n \text{ odd} & c_n = 0 \\ n=2m & c_{2m} = 3c_0\frac{(-1)^m(2m+2)}{(2m+3)!} \end{array}$$

$y(t) = 3c_0 \sum_{m=0}^{\infty} \frac{(-1)^m(2m+2)t^{2m+3}}{(2m+3)!} = 3c_0(\sin t - t\cos t)$.

r=0: One is lead to the equation $0c_3 = 0$ and we can take $c_3 = 0$. Thus

$$\begin{array}{ll} n \text{ odd} & c_n = 0 \\ n=2m & c_{2m} = c_0\frac{(-1)^{m+1}(2m-1)}{(2m)!} \end{array}$$

$y(t) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(2m-1)t^{2m}}{(2m)!} = c_0(t\sin t + \cos t)$.

General Solution: $y = c_1(\sin t - t\cos t) + c_2(t\sin t + \cos t)$.

17. Indicial polynomial: $p(s) = (s-1)^2$; exponents of singularity $s = 1$, multiplicity 2. There is one Frobenius solution.

r = 1 : Let $y(t) = \sum_{n=0}^{\infty} c_n t^{n+1}$. Then

$$n \geq 1 \quad n^2 c_n - n c_{n-1} = 0.$$

This is easy to solve. We get $c_n = \frac{1}{n!}c_0$ and hence

$$y(t) = c_0 \sum_{n=0}^{\infty} \frac{1}{n!} t^{n+1} = c_0 t e^t.$$

Logarithmic Solution: Let $y_1(t) = te^t$. The second independent solution is necessarily of the form

$$y(t) = y_1(t)\ln t + \sum_{n=0}^{\infty} c_n t^{n+1}.$$

Substitution into the differential equation leads to

$$t^2 e^t + \sum_{n=1}^{\infty} (n^2 c_n - n c_{n-1}) t^{n+1} = 0.$$

We write out the power series for $t^2 e^t$ and add corresponding coefficients to get

$$n^2c_n - nc_{n-1} + \frac{1}{(n-1)!}.$$

The following list is a straightforward verification:

$$\begin{array}{ll} n=1 & c_1 = -1 \\ n=2 & c_2 = \frac{-1}{2}\left(1+\frac{1}{2}\right) \\ n=3 & c_3 = \frac{-1}{3!}\left(1+\frac{1}{2}+\frac{1}{3}\right) \\ n=4 & c_4 = \frac{-1}{4!}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right). \end{array}$$

Let $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then an easy argument gives that

$$c_n = \frac{-s_n}{n!}.$$

We now have a second independent solution

$$y_2(t) = te^t \ln t - t\sum_{n=1}^{\infty} \frac{s_n t^n}{n!}.$$

General Solution:

$$y = c_1 te^t + c_2\left(te^t \ln t - t\sum_{n=1}^{\infty} \frac{s_n t^n}{n!}\right).$$

19. Indicial polynomial: $p(s) = (s-2)(s+1)$; exponents of singularity $s = 2$ and $s = -1$.

$$\begin{array}{ll} n=0 & c_0(r-2)(r+1) = 0 \\ n \geq 1 & c_n(n+r-2)(n+r+1) = -c_{n-1}(n+r-1) \end{array}$$

r=2:

$$c_n = 6c_0\frac{(-1)^n(n+1)}{(n+3)!}, \quad n \geq 1$$

$y(t) = 6c_0 \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)t^n}{(n+3)!} = 6c_0\left(\frac{(t+2)e^{-t}}{t} + \frac{t-2}{t}\right)$.

r=-1: The recursion relation becomes $c_n(n-3)(n) = c_{n-1}(n-2) = 0$. Thus

$$\begin{array}{ll} n=1 & c_1 = -\frac{c_0}{2} \\ n=2 & c_2 = 0 \\ n=3 & 0c_3 = 0 \end{array}$$

We can take $c_3 = 0$ and then $c_n = 0$ for all $n \geq 2$. We now have $y(t) = c_0t^{-1}(1-\frac{t}{2}) = \frac{c_0}{2}\left(\frac{2-t}{t}\right)$.

General Solution: $y = c_1\frac{2-t}{t} + c_2\frac{(t+2)e^{-t}}{t}$.

21. Indicial polynomial: $p(s) = s(s-2)$; exponents of singularity $r = 0$ and $r = 2$.

$$\begin{array}{ll} n = 0 & c_0 r(r-2) = 0 \\ n \geq 1 & c_n(n+r)(n+r-2) = -c_{n-1}(n+r-3) \end{array}$$

r=2: The recursion relation becomes $c_n = -\frac{n-1}{n(n+2)} c_{n-1}$. For $n = 1$ we see that $c_1 = 0$ and hence $c_n = 0$ for all $n \geq 1$. It follows that $y(t) = c_0 t^2$ is a solution.

r=0: The recursion relation becomes $c_n(n)(n-2) = -c_{n-1}(n-3) = 0$. Thus

$$\begin{array}{ll} n = 1 & -c_1 = -\frac{c_0}{-2} \\ n = 2 & 0c_2 = -2c_0 \quad \Rightarrow\Leftarrow \end{array}$$

The $n = 2$ case implies an inconsistency in the recursion relation since $c_0 \neq 0$. Since $y_1(t) = t^2$ is a Frobenius solution a second independent solution can be written in the form

$$y(t) = t^2 \ln t + \sum_{n=0}^{\infty} c_n t^n.$$

Substitution leads to

$$t^3 + 2t^2 + (-c_1 - 2c_0)t + \sum_{n=2}^{\infty} (c_n(n)(n-2) + c_{n-1}(n-3))t^n = 0$$

and the following relations:

$$\begin{array}{ll} n = 1 & -c_1 - 2c_0 = 0 \\ n = 2 & 2 - c_1 = 0 \\ n = 3 & 1 + 3c_3 = 0 \\ n \geq 4 & n(n-2)c_n + (n-3)c_{n-3} = 0. \end{array}$$

We now have $c_0 = -1$, $c_1 = 2$, $c_3 = -1/3$. c_2 can be arbitrary so we choose $c_2 = 0$, and $c_n = \frac{-(n-3)c_{n-1}}{n(n-2)}$, for $n \geq 4$. A straightforward calculation gives

$$c_n = \frac{2(-1)^n}{n!(n-2)}.$$

A second independent solution is

$$y_2(t) = t^2 \ln t + \left(-1 + 2t - \frac{t^3}{3} + \sum_{n=4}^{\infty} \frac{2(-1)^n t^n}{n!(n-2)} \right).$$

General Solution: $y = c_1 t^2 + c_2 y_2(t)$.

23. Indicial polynomial: $p(s) = (s^2+1)$; exponents of singularity $r = \pm i$. Let $r = i$ (the case $r = -i$ gives equivalent results). The recursion relation that arises from $y(t) = \sum_{n=0}^{\infty} c_n t^{n+i}$ is $c_n((n+i)^2+1) - c_{n-1}((n-2+i)^2+1) = 0$ and hence

$$c_n = \frac{(n-2)(n-2+2i)}{n(n+2i)} c_{n-1}.$$

A straightforward calculation gives the first few terms as follows:

$$\begin{array}{ll} n=1 & c_1 = \frac{1-2i}{1+2i} c_0 \\ n=2 & c_2 = 0 \\ n=3 & c_3 = 0 \end{array}$$

and hence $c_n = 0$ for all $n \geq 2$. Therefore $y(t) = c_0(t^i + \left(\frac{1-2i}{1+2i}\right) t^{1+i})$. Since $t > 0$ we can write $t^i = e^{i \ln t} = \cos \ln t + i \sin \ln t$, by Euler's formula. Separating the real and imaginary parts we get two independent solutions

$$\begin{aligned} y_1(t) &= -3\cos\ln t - 4\sin\ln t + 5t\cos\ln t \\ y_2(t) &= -3\sin\ln t + 4\cos\ln t + 5t\sin\ln t. \end{aligned}$$

25. Indicial polynomial: $p(s) = (s^2+1)$; exponents of singularity $r = \pm i$. Let $r = i$ (the case $r = -i$ gives equivalent results). The recursion relation that arises from $y(t) = \sum_{n=0}^{\infty} c_n t^{n+i}$ is

$$\begin{array}{ll} n=1 & c_1 = c_0 \\ n \geq 2 & c_n((n+i)^2+1) + c_{n-1}(-2n-2i+1) + c_{n-2} = 0 \end{array}$$

A straightforward calculation gives the first few terms as follows:

$$\begin{array}{ll} n=1 & c_1 = c_0 \\ n=2 & c_2 = \frac{1}{2!} c_0 \\ n=3 & c_3 = \frac{1}{3!} c_0 \\ n=4 & c_4 = \frac{1}{4!} c_0. \end{array}$$

An easy induction argument gives

$$c_n = \frac{1}{n!} c_0.$$

We now get

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} c^n t^{n+i} \\
&= c_0 \sum_{n=0}^{\infty} \frac{t^{n+i}}{n!} \\
&= c_0 t^i e^t.
\end{aligned}$$

Since $t > 0$ we can write $t^i = e^{i \ln t} = \cos \ln t + i \sin \ln t$, by Euler's formula. Now separating the real and imaginary parts we get two independent solutions

$$y_1(t) = e^t \cos \ln t \quad \text{and} \quad y_2(t) = e^t \sin \ln t.$$

Section 7.4

1. Let

$$y(t) = t^{2k+1} \sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} c_n t^{n+2k+1}$$

Then

$$\begin{aligned}
ty''(t) &= \sum_{n=0}^{\infty} (n+2k)(n+2k+1) c_n t^{n+2k} \\
2ity'(t) &= \sum_{n=1}^{\infty} 2i(n+2k) c_{n-1} t^{n+2k} \\
-2ky'(t) &= \sum_{n=0}^{\infty} -2k(n+2k+1) c_n t^{n+2k} \\
-2iky(t) &= \sum_{n=1}^{\infty} -2ik c_{n-1} t^{n+2k}
\end{aligned}$$

By assumption the sum of the series is zero. The $n = 0$ terms in the first and third sum give $(2k)(2k+1)c_0 - 2k(2k+1)c_0 = 0$. Thus we can start all the series at $n = 1$. For $n \geq 1$ we get $n(n+2k+1)c_n + 2i(n+k)c_{n-1} = 0$ which implies

$$c_n = \frac{-2i(n+k)}{n(n+2k+1)} c_{n-1}.$$

Since $c_0 \neq 0$ it follows from this recursion relation that $c_n \neq 0$ for all $n \geq 0$. Therefore the Frobenius solution $y(t)$ is not a polynomial.

3. Since differentiation respects the real and imaginary parts of complex-valued functions we have

$$\begin{aligned}
B_k(t) &= \operatorname{Re}(b_k(t)e^{it}) \\
B_k'(t) &= \operatorname{Re}((b_k(t)e^{it})') = \operatorname{Re}((b_k'(t) + ib_k(t))e^{it}) \\
B_k''(t) &= \operatorname{Re}((b_k''(t) + 2ib_k'(t) - b_k(t))e^{it}).
\end{aligned}$$

It follows now from Proposition 4 that

$$\begin{aligned}
0 &= t^2 B_k''(t) - 2ktB_k'(t) + (t^2 + 2k)B_k(t) \\
&= t^2 \operatorname{Re}(b_k(t)e^{it})'' - 2kt \operatorname{Re}(b_k(t)e^{it})' + (t^2 + 2k) \operatorname{Re}(b_k(t)e^{it}) \\
&= \operatorname{Re}((t^2(b_k''(t) + 2ib_k'(t) - b_k(t)) - 2kt(b_k'(t) + ib_k(t)) + (t^2 + 2k)b_k(t))e^{it}) \\
&= \operatorname{Re}\left((t^2 b_k''(t) + 2t(it - k)b_k'(t) - 2k(it - 1)b_k(t))e^{it}\right).
\end{aligned}$$

Apply Lemma 6 to get

$$t^2 b_k''(t) + 2t(it - k)b_k'(t) - 2k(it - 1)b_k(t) = 0.$$

5. Let $g(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2-1)^k}\right\}$. Then

$$\begin{aligned}
\mathcal{L}\{tg(t)\} &= -\frac{d}{ds}\left(\mathcal{L}\{g(t)\}\right) = -\frac{d}{ds}\left(\frac{s}{(s^2-1)^k}\right) \\
&= -\frac{(s^2-1)^k - 2ks^2(s^2-1)^{k-1}}{(s^2-1)^{2k}} \\
&= \frac{2ks^2 - (s^2-1)}{(s^2-1)^{k+1}} = \frac{(2k-1)(s^2-1) + 2k}{(s^2-1)^{k+1}} \\
&= \frac{2k-1}{(s^2-1)^k} + \frac{2k}{(s^2-1)^{k+1}}.
\end{aligned}$$

Divide by $2k$, solve for the second term in the last line, and apply the inverse Laplace transform to get

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2-1)^{k+1}}\right\} &= \frac{t}{2k}g(t) - \frac{(2k-1)}{2k}\mathcal{L}^{-1}\left\{\frac{1}{(s^2-1)^k}\right\} \\
&= \frac{t}{2k}\mathcal{L}^{-1}\left\{\frac{s}{(s^2-1)^k}\right\} - \frac{(2k-1)}{2k}\mathcal{L}^{-1}\left\{\frac{1}{(s^2-1)^k}\right\}.
\end{aligned}$$

By the definition of C_k and D_k we get

$$\frac{1}{2^k k!}C_k(t) = \frac{t}{2^k k!}D_{k-1}(t) - \frac{2k-1}{2^k k!}C_{k-1}(t).$$

Simplifying gives the result.

7. Multiply the equation in Exercise 5 by t and use the formula in Exercise 4 to get $D_{k+1}(t) = t^2 D_{k-1} - (2k-1)D_k(t)$. Now shift k and the formula follows.

9. By the Input Derivative Principle we have

$$\begin{aligned}
\frac{1}{2^k k!}\mathcal{L}\left\{C_k'(t)\right\} &= \frac{1}{2^k k!}(s\mathcal{L}\left\{C_k(t)\right\} - C_k(0)) \\
&= \frac{s}{(s^2-1)^{k+1}} \\
&= \frac{1}{2^k k!}\mathcal{L}\left\{D_k(t)\right\}.
\end{aligned}$$

Laplace inversion gives the first formula. In a similar way the Input Derivative Principle gives

$$\begin{aligned}
\frac{1}{2^k k!}\mathcal{L}\left\{D_k'(t)\right\} &= \frac{1}{2^k k!}(s\mathcal{L}\left\{D_k(t)\right\} - D_k(0)) \\
&= \frac{s^2}{(s^2-1)^{k+1}} \\
&= \frac{s^2-1}{(s^2-1)^{k+1}} + \frac{1}{(s^2-1)^{k+1}} \\
&= \frac{1}{(s^2-1)^{k}} + \frac{1}{(s^2-1)^{k+1}} \\
&= \frac{1}{2^{k-1}(k-1)!}\mathcal{L}\left\{C_{k-1}(t)\right\} + \frac{1}{2^k k!}\mathcal{L}\left\{C_k(t)\right\}.
\end{aligned}$$

Simplifying and Laplace inversion gives the result.

11. Since $s^2 - 1 = (s-1)(s+1)$ it follows that $\frac{1}{(s^2-1)^{k+1}}$ is an $s-1$-chain and an $s+1$-chain, each of length $k+1$. Hence there are constants α_n and β_n so that

$$\frac{1}{(s^2-1)^{k+1}} = \sum_{n=1}^{k+1} \frac{\alpha_n}{(s-1)^n} + \frac{\beta_n}{(s+1)^n}.$$

Now replace s by $-s$. The left-hand side does not change so we get

$$\begin{aligned}
&\frac{1}{(s^2-1)^{k+1}} \\
&= \sum_{n=1}^{k+1} \frac{\alpha_n}{(-s-1)^n} + \frac{\beta_n}{(-s+1)^n} \\
&= \sum_{n=1}^{k+1} \frac{\alpha_n(-1)^n}{(s+1)^n} + \frac{\beta_n(-1)^n}{(s-1)^n}.
\end{aligned}$$

It follows now by the uniqueness of partial fraction decompositions that

$$\beta_n = (-1)^n \alpha_n.$$

Laplace inversion now gives

$$\begin{aligned} & \mathcal{L}^{-1}\left\{\frac{1}{(s^2-1)^{k+1}}\right\} \\ = & \sum_{n=1}^{k+1} \alpha_n \frac{t^{n-1}}{(n-1)!}e^t + \alpha_n(-1)^n \frac{t^{n-1}}{(n-1)!}e^{-t} \\ = & \sum_{n=1}^{k+1} \alpha_n \frac{t^{n-1}}{(n-1)!}e^t - \alpha_n \frac{(-t)^{n-1}}{(n-1)!}e^{-t} \end{aligned}$$

Let $f(t) = \sum_{n=1}^{k+1} \alpha_n \frac{t^{n-1}}{(n-1)!}$. Then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2-1)^{k+1}}\right\} = f(t)e^t - f(-t)e^{-t}.$$

Up to the constant $2^k k!$, the polynomial $f(t)$ is $c_k(t)$. A similar argument gives the second part of the problem.

13. 1. It is easy to see from the definition of c_k and Exercise 5 that c_k satisfies

$$c_{k+2}(t) = t^2 c_k(t) - (2k+3)c_{k+1}(t)$$

and therefore $c_{k+2}(0) = -(2k+3)c_{k+1}(0)$. An easy check gives $c_1(t) = \frac{t-1}{2}$ and thus $c_1(0) = \frac{-1}{2}$. Recursively we get

$$\begin{aligned} c_1(0) &= \tfrac{-1}{2} & \qquad c_3(0) &= -5c_2(0) = \tfrac{-5\cdot 3}{2} \\ c_2(0) &= -3c_1(0) = \tfrac{3}{2} & \qquad c_4(0) &= 7c_3(0) = \tfrac{7\cdot 5\cdot 3}{2} \end{aligned}$$

Inductively, we get

$$\begin{aligned} c_k(0) &= \frac{(-1)^k}{2}(2k-1)\cdot(2k-3)\cdot(2k-5)\cdots 1 \\ &= \frac{(-1)^k}{2}\frac{(2k)!}{2^k k!} \\ &= \frac{(-1)^k(2k)!}{2^{k+1}k!}. \end{aligned}$$

2. From Exercise 4 it is easy to see that $d_k(t) = tc_{k-1}$ and so $d_k'(0) = c_{k-1}(0) = \frac{(-1)^{k-1}(2(k-1))!}{2^k(k-1)!}$.

15. Merely put the previous calculations together.

Section 8.1

1. $B+C=\begin{bmatrix} 1 & 1 \\ -1 & 7 \\ 0 & 3 \end{bmatrix}$, $B-C=\begin{bmatrix} 1 & -3 \\ 5 & -1 \\ -2 & 1 \end{bmatrix}$, and $2B-3C=\begin{bmatrix} 2 & -8 \\ 13 & -6 \\ -5 & 1 \end{bmatrix}$

3. $A(B+C)=AB+AC=\begin{bmatrix} 3 & 4 \\ 1 & 13 \end{bmatrix}$, $(B+C)A=\begin{bmatrix} 3 & -1 & 7 \\ 3 & 1 & 25 \\ 5 & 0 & 12 \end{bmatrix}$

5. $AB=\begin{bmatrix} 6 & 4 & -1 & -8 \\ 0 & 2 & -8 & 2 \\ 2 & -1 & 9 & -5 \end{bmatrix}$

7. $CA=\begin{bmatrix} 8 & 0 \\ 4 & -5 \\ 8 & 14 \\ 10 & 11 \end{bmatrix}$

9. $ABC=\begin{bmatrix} 8 & 9 & -48 \\ 4 & 0 & -48 \\ -2 & 3 & 40 \end{bmatrix}$.

15. $\begin{bmatrix} 0 & 0 & 1 \\ 3 & -5 & -1 \\ 0 & 0 & 5 \end{bmatrix}$

17. (a) Choose, for example, $A=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.
(b) $(A+B)^2=A^2+2AB+B^2$ precisely when $AB=BA$.

19. $B^n=\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

21. (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A=\begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$; the two rows of A are switched. (b) $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} A=\begin{bmatrix} v_1+cv_2 \\ v_2 \end{bmatrix}$; to the first row is added c times the second row while the second row is unchanged, (c) to the second row is added c times the first row while the first row is unchanged. (d) the first row is multiplied by a while the second row is unchanged, (e) the second row is multiplied by a while the first row is unchanged.

23.

$$
\begin{aligned}
& F(\theta_1)F(\theta_2) \\
= \; & \begin{bmatrix} \cosh\theta_1 & \sinh\theta_1 \\ \sinh\theta_1 & \cosh\theta_1 \end{bmatrix} \begin{bmatrix} \cosh\theta_2 & \sinh\theta_2 \\ \sinh\theta_2 & \cosh\theta_2 \end{bmatrix} \\
= \; & \begin{bmatrix} \cosh\theta_1\cosh\theta_2 + \sinh\theta_1\sinh\theta_2 & \cosh\theta_1\sinh\theta_2 + \sinh\theta_1\cosh\theta_2 \\ \sinh\theta_1\cosh\theta_2 + \cosh\theta_1\sinh\theta_2 & \sinh\theta_1\sinh\theta_2 + \cosh\theta_1\cosh\theta_2 \end{bmatrix} \\
= \; & \begin{bmatrix} \cosh(\theta_1+\theta_2) & \sinh(\theta_1+\theta_2) \\ \sinh(\theta_1+\theta_2) & \cosh(\theta_1+\theta_2) \end{bmatrix} \\
= \; & F(\theta_1+\theta_2),
\end{aligned}
$$

We used the addition formulas for sinh and cosh in the second line.

Section 8.2

1. $A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 6 \end{bmatrix}$, and $[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 1 & 1 & -1 & 4 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 6 \end{array}\right]$.

3.
$$
\begin{array}{rcrcrcrcrcr}
x_1 & - & & & x_3 & + & 4x_4 & + & 3x_5 & = & 2 \\
5x_1 & + & 3x_2 & - & 3x_3 & - & x_4 & - & 3x_5 & = & 1 \\
3x_1 & - & 2x_2 & + & 8x_3 & + & 4x_4 & - & 3x_5 & = & 3 \\
-8x_1 & + & 2x_2 & & & + & 2x_4 & + & x_5 & = & -4
\end{array}
$$

5. RREF

7. $m_2(1/2)(A) = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

9. $t_{1,3}(-3)(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 & -11 & -8 \\ 0 & 1 & 0 & -4 & -2 \\ 0 & 0 & 1 & 9 & 6 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

19. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}, \alpha \in \mathbb{R}$

21. $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$

23. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

25. $\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$

27. $\emptyset$

29. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

31. $\left\{ \begin{bmatrix} -34 \\ -40 \\ 39 \\ 1 \end{bmatrix} \right\}$

33. The equation $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ has solution $a = 2$ and $b = 3$. By Proposition 7 $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ is a solution.

35. If $\mathbf{x}_i$ is the solution set for $A\mathbf{x} = \mathbf{b}_i$ then $\mathbf{x}_1 = \begin{bmatrix} -7/2 \\ 7/2 \\ -3/2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -3/2 \\ 3/2 \\ -1/2 \end{bmatrix}$,

and $\mathbf{x}_3 = \begin{bmatrix} 7 \\ -6 \\ 3 \end{bmatrix}$.

Section 8.3

1. $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

3. not invertible

5. not invertible

7. $\begin{bmatrix} -6 & 5 & 13 \\ 5 & -4 & -11 \\ -1 & 1 & 3 \end{bmatrix}$

9. $\begin{bmatrix} -29 & 39/2 & -22 & 13 \\ 7 & -9/2 & 5 & -3 \\ -22 & 29/2 & -17 & 10 \\ 9 & -6 & 7 & -4 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 \end{bmatrix}$

13. $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

15. $\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{10} \begin{bmatrix} -2 & 4 & 4 \\ -2 & -1 & 4 \\ -6 & 2 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 16 \\ 11 \\ 18 \end{bmatrix}$

17. $\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{2} \begin{bmatrix} -58 & 39 & -44 & 26 \\ 14 & -9 & 10 & -6 \\ -44 & 29 & -34 & 20 \\ 18 & -12 & 14 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 19 \\ -4 \\ 15 \\ -6 \end{bmatrix}$

19. $(A^t)^{-1} = (A^{-1})^t$

21. $F(\theta)^{-1} = F(-\theta)$

Section 8.4

1. 1

3. 10

5. -21

7. 2

9. 0

11. $\frac{1}{s^2-6s+8}\begin{bmatrix} s-3 & 1 \\ 1 & s-3 \end{bmatrix}$ $s = 2, 4$

13. $\frac{1}{(s-1)^3}\begin{bmatrix} (s-1)^2 & 3 & s-1 \\ 0 & (s-1)^2 & 0 \\ 0 & 3(s-1) & (s-1)^2 \end{bmatrix}$ $s = 1$

15. $\frac{1}{s^3+s^2+4s+4}\begin{bmatrix} s^2+s & 4s+4 & 0 \\ -s-1 & s^2+s & 0 \\ s-4 & 4s+4 & s^2+4 \end{bmatrix}$ $s = -1, \pm 2i$

17. no inverse

19. $\frac{1}{8}\begin{bmatrix} 4 & -4 & 4 \\ -1 & 3 & -1 \\ -5 & -1 & 3 \end{bmatrix}$

21. $\frac{1}{6}\begin{bmatrix} 2 & -98 & 9502 \\ 0 & 3 & -297 \\ 0 & 0 & 6 \end{bmatrix}$

23. $\frac{1}{15}\begin{bmatrix} 55 & -95 & 44 & -171 \\ 50 & -85 & 40 & -150 \\ 70 & -125 & 59 & -216 \\ 65 & -115 & 52 & -198 \end{bmatrix}$

25. $\det A = 1$, $\det A(1, \mathbf{b}) = \det \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = 5$, and $\det A(2, \mathbf{b}) = \det \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} = -3$. It follows that $x_1 = 5/1 = 5$ and $x_2 = -3/1 = -3$

27. $\det A = -10$, $\det A(1,\mathbf{b}) = \det\begin{bmatrix} -2 & 0 & -2 \\ 1 & -2 & 0 \\ 2 & 2 & -1 \end{bmatrix} = -16$, $\det A(2,\mathbf{b}) = \det\begin{bmatrix} 1 & -2 & -2 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} = -11$, and $\det A(3,\mathbf{b}) = \det\begin{bmatrix} 1 & 0 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = -18$. It follows that $x_1 = 16/10$, $x_2 = 11/10$, and $x_3 = 18/10$.

Section 8.5

1. The characteristic polynomial is $c_A(s) = (s-1)(s-2)$. The eigenvalues are thus $s = 1, 2$. The eigenspaces are $E_1 = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and $E_2 = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

3. The characteristic polynomial is $c_A(s) = s^2 - 2s + 1 = (s-1)^2$. The only eigenvalue is $s = 1$. The eigenspace is $E_1 = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

5. The characteristic polynomial is $c_A(s) = s^2 + 2s - 3 = (s+3)(s-1)$. The eigenvalues are thus $s = -3, 1$. The eigenspaces are $E_{-3} = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ and $E_1 = \text{Span}\left\{\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}$.

7. The characteristic polynomial is $c_A(s) = s^2 + 2s + 10 = (s+1)^2 + 3^2$. The eigenvalues are thus $s = -1 \pm 3i$. The eigenspaces are $E_{-1+3i} = \text{Span}\left\{\begin{bmatrix} 7+i \\ 10 \end{bmatrix}\right\}$ and $E_{-1-3i} = \text{Span}\left\{\begin{bmatrix} 7-i \\ 10 \end{bmatrix}\right\}$.

9. The eigenvalues are $s = -2, 3$. $E_{-2} = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$, $E_3 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$,

11. The eigenvalues are $s = 0, 2, 3$. $E_0 = \text{NS}(A) = \text{Span}\left\{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$, $E_2 = \text{Span}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}$, $E_3 = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

13. We write $c_A(s) = (s-2)((s-2)^2+1)$ to see that the eigenvalues are $s = 2, 2 \pm i$. $E_2 = \text{Span}\left\{\begin{bmatrix}2\\3\\1\end{bmatrix}\right\}$, $E_{2+i} = \text{Span}\left\{\begin{bmatrix}-4+3i\\4+2i\\5\end{bmatrix}\right\}$, $E_{2-i} = \text{Span}\left\{\begin{bmatrix}-4-3i\\4-2i\\5\end{bmatrix}\right\}$

Section 9.2

1. nonlinear, because of the presence of the product y_1y_2.

3. We may write the system in the form

$$\boldsymbol{y}' = \begin{bmatrix}\sin t & 0\\ 1 & \cos t\end{bmatrix}\boldsymbol{y}.$$

It is linear and homogeneous, but not constant coefficient.

5. We write the system in the form

$$\boldsymbol{y}' = \begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 1 & 2 & 0\end{bmatrix}\boldsymbol{y}.$$

It is linear, constant coefficient, and homogeneous.

7. First note that $y_1(0) = 0$ and $y_2(0) = 1$, so the initial condition is satisfied. Then

$$\boldsymbol{y}'(t) = \begin{bmatrix}y_1'(t)\\ y_2'(t)\end{bmatrix} = \begin{bmatrix}e^t - 3e^{3t}\\ 2e^t - 3e^{3t}\end{bmatrix}$$

while

$$\begin{aligned}\begin{bmatrix}5 & -2\\ 4 & -1\end{bmatrix}\boldsymbol{y}(t) &= \begin{bmatrix}5(e^t - e^{3t}) - 2(2e^t - e^{3t})\\ 4(e^t - e^{3t}) - (2e^t - e^{3t})\end{bmatrix}\\ &= \begin{bmatrix}e^t - 3e^{3t}\\ 2e^t - 3e^{3t}\end{bmatrix}.\end{aligned}$$

Thus $\boldsymbol{y}'(t) = \begin{bmatrix}5 & -2\\ 4 & -1\end{bmatrix}\boldsymbol{y}(t)$, as required.

9. First note that $y_1(0) = 1$ and $y_2(0) = 3$, so the initial condition is satisfied. Then

$$\boldsymbol{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -e^{-t} + e^t + te^t \\ -3e^{-t} + e^t + te^t \end{bmatrix}$$

while

$$\begin{aligned} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} + te^t \\ 3e^{-t} + te^t \end{bmatrix} + \begin{bmatrix} e^t \\ e^t \end{bmatrix} &= \begin{bmatrix} 2e^{-t} + 2te^t - 3e^{-t} - te^t + e^t \\ 3e^{-t} + 3te^t - 6e^{-t} - 2te^t + e^t \end{bmatrix} \\ &= \begin{bmatrix} -e^{-t} + te^t + e^t \\ -3e^{-t} + te^t + e^t \end{bmatrix}. \end{aligned}$$

Thus $\boldsymbol{y}'(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \boldsymbol{y}(t) + \begin{bmatrix} e^t \\ e^t \end{bmatrix}$, as required.

In solutions, 11–15, $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$.

11. Let $y_1 = y$ and $y_2 = y'$. Then $y_1' = y' = y_2$ and $y_2' = y'' = -5y' - 6y + e^{2t} = -6y_1 - 5y_2 + e^{2t}$. Letting $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this can be expressed in vector form as

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

13. Let $y_1 = y$ and $y_2 = y'$. Then $y_1' = y' = y_2$ and $y_2' = y'' = k^2 y + A\cos\omega t = k^2 y_1 + A\cos\omega t$. Letting $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this can be expressed in vector form as

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ k^2 & 0 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 0 \\ A\cos\omega t \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

15. Let $y_1 = y$ and $y_2 = y'$. Then $y_1' = y' = y_2$ and $y_2' = y'' = -\frac{2}{t}y' - \frac{1}{t^2}y = -\frac{1}{t^2}y_1 - \frac{2}{t}y_2$. Letting $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this can be expressed in vector form as

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -\frac{1}{t^2} & -\frac{2}{t} \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(1) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

17. $A'(t) = \begin{bmatrix} -3e^{-3t} & 1 \\ 2t & 2e^{2t} \end{bmatrix}$

19. $\boldsymbol{y}'(t) = \begin{bmatrix} 1 \\ 2t \\ t^{-1} \end{bmatrix}$

21. $\boldsymbol{v}'(t) = \begin{bmatrix} -2e^{-2t} & \frac{2t}{t^2+1} & -3\sin 3t \end{bmatrix}$

23. $\frac{1}{4}\begin{bmatrix} e^2 - e^{-2} & e^2 + e^{-2} - 2 \\ 2 - e^2 - e^{-2} & e^2 - e^{-2} \end{bmatrix}$

25. $\begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix}$

27. $\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ \frac{2}{s^3} & \frac{1}{s-2} \end{bmatrix}$

29. $\begin{bmatrix} \frac{3!}{s^4} & \frac{2s}{(s^2+1)^2} & \frac{1}{(s+1)^2} \\ \frac{2-s}{s^3} & \frac{s-3}{s^2-6s+13} & \frac{3}{s} \end{bmatrix}$

31. $\frac{2}{s^2-1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 1 & 2t & 3t^2 \end{bmatrix}$

35. We have $\begin{bmatrix} \dfrac{2s}{s^2-1} & \dfrac{2}{s^2-1} \\ \dfrac{2}{s^2-1} & \dfrac{2s}{s^2-1} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} + \dfrac{1}{s-1} & \dfrac{-1}{s+1} + \dfrac{1}{s-1} \\ \dfrac{-1}{s+1} + \dfrac{1}{s-1} & \dfrac{1}{s+1} + \dfrac{1}{s-1} \end{bmatrix}$. Laplace inversion gives $\begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}$

Section 9.3

1.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \\ A^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \\ A^3 &= \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix} \\ &\vdots \\ A^n &= \begin{bmatrix} 1 & 0 \\ 0 & (-2)^n \end{bmatrix}. \end{aligned}$$

It follows now that

$$\begin{aligned}
e^{At} &= I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & -2t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} t^2 & 0 \\ 0 & (-2t)^2 \end{bmatrix} + \cdots + \frac{1}{n!}\begin{bmatrix} t^n & 0 \\ 0 & (-2t)^n \end{bmatrix} + \cdots \\
&= \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}.
\end{aligned}$$

3.

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
A^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \\
A^3 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A
\end{aligned}$$

It follows now that $A^n = I$ if n is even and $A^n = A$ if n is odd. Thus

$$\begin{aligned}
e^{At} &= I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots \\
&= I + At + I\frac{t^2}{2!} + A\frac{t^3}{3!} + I\frac{t^4}{4!} + A\frac{t^5}{5!} + \cdots \\
&= I\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + A\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) \\
&= I\cosh t + A\sinh t \\
&= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}.
\end{aligned}$$

5.

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
A^2 &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\
A^3 &= \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \\
&\vdots \\
A^n &= \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}.
\end{aligned}$$

It follows now that

$$\begin{aligned} e^{At} &= I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & t \\ t & t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2t^2 & 2t^2 \\ 2t^2 & 2t^2 \end{bmatrix} + \cdots + \frac{1}{n!}\begin{bmatrix} 2^{n-1}t^n & 2^{n-1}t^n \\ 2^{n-1}t^n & 2^{n-1}t^n \end{bmatrix} + \cdots \end{aligned}$$

The $(1,1)$ entry is

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{2^{n-1}t^n}{n!} &= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2t)^n}{n!} \\ &= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \\ &= \frac{1}{2} + \frac{1}{2}e^{2t} \end{aligned}$$

The $(1,2)$ entry is

$$\begin{aligned} 0 + \sum_{n=1}^{\infty} \frac{2^{n-1}t^n}{n!} &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2t)^n}{n!} \\ &= -\frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \\ &= -\frac{1}{2} + \frac{1}{2}e^{2t} \end{aligned}$$

Since the $(1,1)$ entry and the $(2,2)$ entry are equal and the $(1,2)$ entry and the $(2,1)$ entry are equal we have

$$e^{At} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{2t} & -\frac{1}{2} + \frac{1}{2}e^{2t} \\ -\frac{1}{2} + \frac{1}{2}e^{2t} & \frac{1}{2} + \frac{1}{2}e^{2t} \end{bmatrix}$$

7.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ A^2 &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\ A^3 &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix} \\ A^4 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix} \\ A^5 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 32 \end{bmatrix} \end{aligned}$$

The $(1,1)$ entry and the $(2,2)$ entry of e^{At} are equal and are

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots .$$

The $(1,2)$ entry and the $(2,1)$ entry of e^{At} have opposite signs. The $(2,1)$ entry is

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots .$$

The $(3,3)$ entry is

$$e^{2t} = 1 + 2t + \frac{(2t)^2}{2!} + \cdots .$$

All other entries are zero thus

$$e^{At} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

9. The characteristic polynomial is $c_A(s) = s(s-3)$ and $sI - A = \begin{bmatrix} s-1 & 1 \\ 2 & s-2 \end{bmatrix}$. Thus $(sI-A)^{-1} = \begin{bmatrix} \frac{s-2}{s(s-3)} & \frac{-1}{s(s-3)} \\ \frac{-2}{s(s-3)} & \frac{s-1}{s(s-3)} \end{bmatrix}$. A partial fraction decomposition of each entry gives

$$(sI - A)^{-1} = \frac{1}{s} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} + \frac{1}{s-3} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

Thus $e^{At} = \mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} \frac{2}{3}+\frac{1}{3}e^{3t} & \frac{1}{3}-\frac{1}{3}e^{3t} \\ \frac{2}{3}-\frac{2}{3}e^{3t} & \frac{1}{3}+\frac{2}{3}e^{3t} \end{bmatrix}$

11. The characteristic polynomial is $c_A(s) = (s-3)(s+1)+5 = s^2-2s+2 = (s-1)^2+1$ and $sI-A = \begin{bmatrix} s-3 & -5 \\ 1 & s+1 \end{bmatrix}$. Thus the resolvent matrix is $(sI-A)^{-1} = \begin{bmatrix} \frac{s+1}{(s-1)^2+1} & \frac{5}{(s-1)^2+1} \\ \frac{-1}{(s-1)^2+1} & \frac{s-3}{(s-1)^2+1} \end{bmatrix}$ which we write as

$$(sI-A)^{-1} = \begin{bmatrix} \frac{s-1}{(s-1)^2+1} & \frac{5}{(s-1)^2+1} \\ \frac{-1}{(s-1)^2+1} & \frac{s-1}{(s-1)^2+1} \end{bmatrix} + \begin{bmatrix} \frac{2}{(s-1)^2+1} & 0 \\ 0 & \frac{-2}{(s-1)^2+1} \end{bmatrix}.$$

Therefore

$$e^{At} = \begin{bmatrix} e^t\cos t + 2e^t\sin t & 5e^t\sin t \\ -e^t\sin t & e^t\cos t - 2e^t\sin t \end{bmatrix}$$

13. The characteristic polynomial is $c_A(s) = s^3$ and $sI-A = \begin{bmatrix} s & -1 & -1 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix}$.

Thus $(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{s+1}{s^3} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$ and $e^{At} = \begin{bmatrix} 1 & t & t+\frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$

15. Let $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $N = 2$. Then by Example 6 $e^{Mt} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Thus

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{Mt} & 0 \\ 0 & e^{Nt} \end{bmatrix} \\ &= \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}. \end{aligned}$$

Section 9.4

1. The characteristic matrix and characteristic polynomial are

$$sI-A = \begin{bmatrix} s-2 & 1 \\ -1 & s \end{bmatrix} \quad \text{and} \quad c_A(s) = s^2-2s+1 = (s-1)^2.$$

The characteristic polynomial has root 1 with multiplicity 2.

The standard basis of $\mathcal{E}_{c_A}$ is $\mathcal{B}_{c_A} = \{e^t,\, te^t\}$. It follows that

$$e^{At} = Me^t + Nte^t.$$

Differentiating we obtain

$$\begin{aligned} Ae^{At} &= Me^t + N(e^t + te^t) \\ &= (M+N)e^t + Nte^t. \end{aligned}$$

Now, evaluate each equation at $t = 0$ to obtain:

$$\begin{aligned} I &= M \\ A &= M + N. \end{aligned}$$

from which we get

$$\begin{aligned} M &= I \\ N &= A - I. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} = & \quad Ie^t + (A - I)te^t \\ = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^t + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} te^t \\ = & \begin{bmatrix} e^t + te^t & -te^t \\ te^t & e^t - te^t \end{bmatrix} \end{aligned}$$

3. $c_A(s) = (s-2)(s+2) + 4 = s^2$. Thus $\mathcal{B}_{c_A} = \{1,\, t\}$ and Fulmer's method gives

$$e^{At} = M_1 + M_2 t.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} M_1 &= I \\ M_2 &= A. \end{aligned}$$

Thus

$$e^{At} = \begin{bmatrix} 1 + 2t & t \\ -4t & 1 - 2t \end{bmatrix}.$$

5. The characteristic polynomial is $c_A(s) = s^2 - 2s + 2 = (s-1)^2 + 1$. The standard basis of $\mathcal{E}_{c_A}$ is $\mathcal{B}_{c_A} = \{e^t \cos t,\, e^t \sin t\}$. It follows that

$$e^{At} = Me^t \cos t + Ne^t \sin t.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M \\ A &= M + N. \end{aligned}$$

from which we get

$$\begin{aligned} M &= I \\ N &= A - I. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= Ie^t \cos t + (A - I)e^t \sin t \\ &= \begin{bmatrix} e^t \cos t & 0 \\ 0 & e^t \cos t \end{bmatrix} + \begin{bmatrix} 3e^t \sin t & -10e^t \sin t \\ e^t \sin t & -3e^t \sin t \end{bmatrix} \\ &= \begin{bmatrix} e^t \cos t + 3e^t \sin t & -10e^t \sin t \\ e^t \sin t & e^t \cos t - 3e^t \sin t \end{bmatrix} \end{aligned}$$

7. The characteristic polynomial is $c_A(s) = s^2 - 4$ and has roots $-2, 2$. The standard basis of $\mathcal{E}_{c_A}$ is $\mathcal{B}_{c_A} = \{e^{2t}, e^{-2t}\}$. It follows that

$$e^{At} = Me^{2t} + Ne^{-2t}.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M + N \\ A &= 2M - 2N. \end{aligned}$$

from which we get

$$\begin{aligned} M &= \frac{1}{4}(A + 2I) \\ N &= -\frac{1}{4}(A - 2I). \end{aligned}$$

Thus,

$$
\begin{aligned}
e^{At} &= \frac{1}{4}(A+2I)e^{2t} - \frac{1}{4}(A-2I)e^{-2t} \\
&= \frac{1}{4}\begin{bmatrix} -7 & 11 \\ -7 & 11 \end{bmatrix} e^{2t} - \frac{1}{4}\begin{bmatrix} -11 & 11 \\ -7 & 7 \end{bmatrix} e^{-2t} \\
&= \frac{1}{4}\begin{bmatrix} -7e^{2t} + 11e^{-2t} & 11e^{2t} - 11e^{-2t} \\ -7e^{2t} + 7e^{-2t} & 11e^{2t} - 7e^{-2t} \end{bmatrix}
\end{aligned}
$$

9. The characteristic polynomial is $c_A(s) = s^2 - 4s + 13 = ((s-2)^2 + 3^2)$. The standard basis of $\mathcal{E}_{c_A}$ is $\mathcal{B}_{c_A} = \left\{e^{2t}\cos 3t,\, e^{2t}\sin 3t\right\}$. It follows that

$$e^{At} = Me^{2t}\cos 3t + Ne^{2t}\sin 3t.$$

Differentiating and evaluating at $t = 0$ gives

$$
\begin{aligned}
I &= M \\
A &= 2M + 3N.
\end{aligned}
$$

from which we get

$$
\begin{aligned}
M &= I \\
N &= \frac{1}{3}(A - 2I) = \begin{bmatrix} 8 & 13 \\ -5 & -8 \end{bmatrix}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
e^{At} &= Ie^{2t}\cos 3t + \frac{1}{3}(A - I)e^{2t}\sin 3t \\
&= \begin{bmatrix} e^{2t}\cos 3t & 0 \\ 0 & e^{2t}\cos 3t \end{bmatrix} + \begin{bmatrix} 8e^{2t}\sin 3t & 13e^{2t}\sin 3t \\ -5e^{2t}\sin 3t & -8e^{2t}\sin 3t \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}\cos 3t + 8e^{2t}\sin 3t & 13e^{2t}\sin 3t \\ -5e^{2t}\sin 3t & e^{2t}\cos 3t - 8e^{2t}\sin 3t \end{bmatrix}
\end{aligned}
$$

11. The characteristic polynomial is $c_A(s) = (s+2)^2$. The standard basis is $\mathcal{B}_{c_A} = \left\{e^{-2t},\, te^{-2t}\right\}$. It follows that

$$e^{At} = Me^{-2t} + Nte^{-2t}.$$

Differentiating and evaluating at $t = 0$ gives

$$
\begin{aligned}
I &= M \\
A &= -2M + N.
\end{aligned}
$$

from which we get

$$\begin{aligned} M &= I \\ N &= A + 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= Ie^{-2t} + (A+2I)te^{-2t} \\ &= \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix} + \begin{bmatrix} -te^{-2t} & te^{-2t} \\ -te^{-2t} & te^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} - te^{-2t} & te^{-2t} \\ -te^{-2t} & e^{-2t} + te^{-2t} \end{bmatrix} \end{aligned}$$

13. In this case $\mathcal{B}_{c_A} = \{1, e^t, e^{-t}\}$. It follows that

$$e^{At} = M + Ne^t + Pe^{-t}.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} M + N + P &= I \\ N - P &= A \\ N + P &= A^2 \end{aligned}$$

from which we get

$$\begin{aligned} M &= I - A^2 \\ N &= \frac{A^2 + A}{2} \\ P &= \frac{A^2 - A}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= M + Ne^t + Pe^{-t} \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} e^t + \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix} e^{-t} \\ &= \begin{bmatrix} 2 - e^{-t} & 0 & -1 + e^{-t} \\ 0 & e^t & 0 \\ 2 - 2e^{-t} & 0 & -1 + 2e^{-t} \end{bmatrix} \end{aligned}$$

15. The standard basis of $\mathcal{E}_{c_A}$ is

$$\mathcal{B}_{c_A} = \left\{ e^t, e^t \cos t, e^t \sin t \right\}.$$

Therefore

$$e^{At} = Me^t + Ne^t \sin t + Pe^t \cos t.$$

Differentiating twice and simplifying we get the system:

$$\begin{aligned} e^{At} &= Me^t + Ne^t \sin t + Pe^t \cos t \\ Ae^{At} &= Me^t + (N-P)e^t \sin t + (N+P)e^t \cos t \\ A^2e^{At} &= Me^t - 2Pe^t \sin t + 2Ne^t \cos t. \end{aligned}$$

Now evaluating at $t = 0$ gives

$$\begin{aligned} I &= M + P \\ A &= M + N + P \\ A^2 &= M + 2N \end{aligned}$$

and solving gives

$$\begin{aligned} N &= A - I \\ M &= A^2 - 2A + 2I \\ P &= -A^2 + 2A - I. \end{aligned}$$

A straightforward calculation gives $A^2 = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 2 & 0 & -2 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$ and

$$N = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \quad M = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad P = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Hence,

$$\begin{aligned} e^{At} &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} e^t + \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} e^t \sin t + \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} e^t \cos t \\ &= \frac{1}{2} \begin{bmatrix} e^t + e^t \cos t & -e^t \sin t & e^t - e^t \cos t \\ 2e^t \sin t & 2e^t \cos t & -2e^t \sin t \\ e^t - e^t \cos t & e^t \sin t & e^t + e^t \cos t \end{bmatrix}. \end{aligned}$$

17. In this case $\mathcal{B}_{c_A} = \{e^t, \cos 2t, \sin 2t\}$. It follows that

$$e^{At} = Me^t + N \cos 2t + P \sin 2t.$$

Differentiating twice and evaluating at $t = 0$ gives

$$\begin{aligned} M + N &= I \\ M + 2P &= A \\ M - 4N &= A^2 \end{aligned}$$

from which we get

$$\begin{aligned} M = \quad & \frac{A^2 + 4I}{5} = & \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix} \\ N = \quad & \frac{I - A^2}{5} = & \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \\ P = \quad & \frac{-A^2 + 5A - 4I}{10} = & \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= Me^t + N\cos 2t + P\sin 2t \\ &= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix} e^t + \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \cos 2t + \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \sin 2t \\ &= \begin{bmatrix} -e^t + 2\cos 2t & -\sin 2t & e^t - \cos 2t \\ 2\sin 2t & \cos 2t & -\sin 2t \\ -2e^t + 2\cos 2t & -\sin 2t & 2e^t - \cos 2t \end{bmatrix} \end{aligned}$$

19. In this case $\mathcal{B}_{c_A} = \{\cos t, \sin t, t\cos t, t\sin t\}$. It follows that

$$e^{At} = M\cos t + N\sin t + Pt\cos t + Qt\sin t.$$

Differentiating three times and evaluating at $t = 0$ gives

$$\begin{aligned} M &= I \\ N + P &= A \\ -M + 2Q &= A^2 \\ -N - 3P &= A^3 \end{aligned}$$

from which we get

$$M = \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N = \frac{A(A^2+3I)}{2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$P = \frac{-A(A^2+I)}{2} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$Q = \frac{A^2+I}{2} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} e^{At} &= M\cos t + N\sin t + Pt\cos t + Qt\sin t \\ &= \begin{bmatrix} \cos t - t\cos t & \sin t - t\sin t & t\cos t & t\sin t \\ -\sin t + t\sin t & \cos t - t\cos t & -t\sin t & t\cos t \\ -t\cos t & -t\sin t & \cos t + t\cos t & \sin t + t\sin t \\ t\sin t & -t\cos t & -\sin t - t\sin t & \cos t + t\cos t \end{bmatrix}. \end{aligned}$$

21. The standard basis is $\mathcal{B}_{c_A} = \{e^{rt}, te^{rt}\}$ so that $e^{At} = Me^{rt} + Nte^{rt}$. Fulmer's method gives

$$\begin{aligned} I &= M \\ A &= rM + N \end{aligned}$$

which are easily solved to give

$$M = I \qquad \text{and} \qquad N = (A - rI).$$

Hence,

$$e^{At} = Ie^{rt} + (A - rI)te^{rt} = (I + (A - rI)t)\, e^{rt}. \tag{1}$$

Section 9.5

1. It is easy to see that $e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix}$. Thus

$$\boldsymbol{y}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -2e^{3t} \end{bmatrix}.$$

3. The characteristic polynomial is $c_A(s) = (s-2)^2$. Thus $\mathcal{B}_{c_A} = \left\{e^{2t}, te^{2t}\right\}$ and

$$e^{At} = M_1 e^{2t} + M_2 te^{2t}.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M_1 \\ A &= 2M_1 + M_2. \end{aligned}$$

and hence $M_1 = I$ and $M_2 = A - 2I$. We thus get $e^{At} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$ and $\boldsymbol{y}(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -e^{2t} + 2te^{2t} \\ 2e^{2t} \end{bmatrix}$

5. The characteristic polynomial is $c_A(s) = s^2 - 1 = (s+1)(s-1)$. Thus $\mathcal{B}_{c_A} = \{e^{-t}, e^t\}$ and

$$e^{At} = M_1 e^{-t} + M_2 e^t.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M_1 + M_2 \\ A &= -M_1 + M_2. \end{aligned}$$

and hence $M_1 = \frac{1}{2}(-A + I)$ and $M_2 = \frac{1}{2}(A + I)$. We thus get $e^{At} = \frac{1}{2}\begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix}$ and

$$\boldsymbol{y}(t) = \frac{1}{2}\begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}.$$

7. The characteristic polynomial is $c_A(s) = (s-1)^2$. Thus $\mathcal{B}_{c_A} = \{e^t, te^t\}$ and

$$e^{At} = M_1 e^t + M_2 te^t.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M_1 \\ A &= M_1 + M_2. \end{aligned}$$

and hence $M_1 = I$ and $M_2 = A - I$. We thus get $e^{At} = \begin{bmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t - 2te^t \end{bmatrix}$ and $\boldsymbol{y}(t) = \begin{bmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t - 2te^t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t - 2te^t \\ e^t - te^t \end{bmatrix}$

9. The characteristic polynomial is $c_A(s) = (s+1)(s^2+4)$. Thus $\mathcal{B}_{c_A} = \{e^{-t}, \cos 2t, \sin 2t\}$ and

$$e^{At} = M_1 e^{-t} + M_2 \cos 2t + M_3 \sin 2t.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M_1 + M_2 \\ A &= -M_1 + 2M_3 \\ A^2 &= M_1 - 4M_2. \end{aligned}$$

and hence

$$\begin{aligned} M_1 &= \frac{1}{5}(A^2 + 4I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ M_2 &= -\frac{1}{5}(A^2 - I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ M_3 &= \frac{1}{10}(A^2 + 5A + 4I) = \begin{bmatrix} 0 & 2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}. \end{aligned}$$

We thus get $e^{At} = \begin{bmatrix} \cos 2t & 2\sin 2t & 0 \\ -\frac{1}{2}\sin 2t & \cos 2t & 0 \\ -e^{-t} + \cos 2t & 2\sin 2t & e^{-t} \end{bmatrix}$ and hence

$$\boldsymbol{y}(t) = \begin{bmatrix} \cos 2t & 2\sin 2t & 0 \\ -\frac{1}{2}\sin 2t & \cos 2t & 0 \\ -e^{-t} + \cos 2t & 2\sin 2t & e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\cos 2t + 2\sin 2t \\ \cos 2t - \sin 2t \\ 2\cos 2t + 2\sin 2t \end{bmatrix}$$

11. A straightforward calculation gives

$$e^{At} = e^{-t}\cos 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t}\sin 2t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

It follows that

$$\begin{aligned}\boldsymbol{y}_h(t) &= e^{At}\boldsymbol{y}_0 \\ &= \left(e^{-t}\cos 2t\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + e^{-t}\sin 2t\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix} \\ &= \begin{bmatrix}e^{-t}\cos 2t\\ -e^{-t}\sin 2t\end{bmatrix}.\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{y}_p &= e^{At} * f(t) \\ &= \left(e^{-t}\cos 2t\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + e^{-t}\sin 2t\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right) * \begin{bmatrix}5\\0\end{bmatrix} \\ &= e^{-t}\cos 2t * 1\begin{bmatrix}5\\0\end{bmatrix} + e^{-t}\sin 2t * 1\begin{bmatrix}0\\-5\end{bmatrix} \\ &= (1 - e^{-t}\cos 2t + 2e^{-t}\sin 2t)\begin{bmatrix}1\\0\end{bmatrix} + (2 - 2e^{-t}\cos 2t - e^{-t}\sin 2t)\begin{bmatrix}0\\-1\end{bmatrix} \\ &= \begin{bmatrix}1 - e^{-t}\cos 2t + 2e^{-t}\sin 2t\\ -2 + 2e^{-t}\cos 2t + e^{-t}\sin 2t\end{bmatrix}.\end{aligned}$$

It now follows that

$$\begin{aligned}\boldsymbol{y}(t) &= \boldsymbol{y}_h + \boldsymbol{y}_p \\ &= \begin{bmatrix}1 + 2e^{-t}\sin 2t\\ -2 + 2e^{-t}\cos 2t\end{bmatrix}\end{aligned}$$

13. A straightforward calculation gives

$$e^{At} = e^{-t}\cos 2t\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + e^{-t}\sin 2t\begin{bmatrix}0 & -2\\ 1/2 & 0\end{bmatrix}$$

It follows that

$$\begin{aligned}\boldsymbol{y}_h(t) &= e^{At}\boldsymbol{y}_0 \\ &= \left(e^{-t}\cos 2t\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + e^{-t}\sin 2t\begin{bmatrix}0 & -2\\ 1/2 & 0\end{bmatrix}\right)\begin{bmatrix}2\\-1\end{bmatrix} \\ &= e^{-t}\cos 2t\begin{bmatrix}2\\-1\end{bmatrix} + e^{-t}\sin 2t\begin{bmatrix}2\\1\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{y}_p &= e^{At} * f(t) \\
&= \left(e^{-t}\cos 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t}\sin 2t \begin{bmatrix} 0 & -2 \\ 1/2 & 0 \end{bmatrix}\right) * \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\
&= ((e^{-t}\cos 2t) * 1) \begin{bmatrix} 4 \\ 1 \end{bmatrix} + ((e^{-t}\sin 2t) * 1) \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\
&= \frac{1}{5}\left(1 - e^{-t}\cos 2t + 2e^{-t}\sin 2t\right) \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \frac{1}{5}\left(2 - 2e^{-t}\cos 2t - e^{-t}\sin 2t\right) \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2e^{-t}\sin 2t \\ 1 - e^{-t}\cos 2t \end{bmatrix}.
\end{aligned}$$

It now follows that

$$\begin{aligned}
\boldsymbol{y}(t) &= \boldsymbol{y}_h + \boldsymbol{y}_p \\
&= e^{-t}\cos 2t \begin{bmatrix} 2 \\ -1 \end{bmatrix} + e^{-t}\sin 2t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2e^{-t}\sin 2t \\ 1 - e^{-t}\cos 2t \end{bmatrix} \\
&= \begin{bmatrix} 2e^{-t}\cos 2t + 4e^{-t}\sin 2t \\ 1 + e^{-t}\sin 2t - 2e^{-t}\cos 2t \end{bmatrix}.
\end{aligned}$$

15. A straightforward calculation gives

$$e^{At} = e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}$$

It follows that

$$\begin{aligned}
\boldsymbol{y}_h(t) &= e^{At}\boldsymbol{y}_0 \\
&= \left(e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2te^t \\ e^t - 4te^t \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{y}_p &= e^{At} * f(t) \\
&= \left(e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}\right) * t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= e^t * t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= (e^t - t - 1) \begin{bmatrix} 1 \\ -2 \end{bmatrix}.
\end{aligned}$$

It now follows that

$$\begin{aligned}\boldsymbol{y}(t) &= \boldsymbol{y}_h + \boldsymbol{y}_p \\ &= \begin{bmatrix} 2te^t \\ e^t - 4te^t \end{bmatrix} + (e^t - t - 1)\begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 2te^t + e^t - t - 1 \\ -4te^t - e^t + 2t + 2 \end{bmatrix}\end{aligned}$$

17. A straightforward calculation gives

$$e^{At} = e^t \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} + te^t \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Clearly

$$\boldsymbol{y}_h(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

while

$$\begin{aligned}\boldsymbol{y}_p(t) &= e^{At} * f(t) \\ &= \left(e^t \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} + te^t \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} \right) * e^{2t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= (e^t * e^{2t}) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + (te^t * e^{2t}) \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} + (e^{2t} * e^{2t}) \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \\ &= (e^{2t} - e^t) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + (e^{2t} - te^t - e^t) \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} + te^{2t} \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} te^t \\ 2te^{2t} - e^{2t} + e^t \\ -2te^{2t} + te^t \end{bmatrix}\end{aligned}$$

It now follows that

$$\boldsymbol{y}(t) = \boldsymbol{y}_h + \boldsymbol{y}_p = \begin{bmatrix} te^t \\ 2te^{2t} - e^{2t} + e^t \\ -2te^{2t} + te^t \end{bmatrix}$$

19. y_1 and y_2 are related to each other as follows:

$$\begin{aligned} y_1' &= 2 - 2y_1 \\ y_2' &= 2y_1 - y_2 \end{aligned}$$

with initial conditions $y_1(0) = 4$ and $y_2(0) = 0$. Let $A = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}$, $\boldsymbol{f}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and $\boldsymbol{y}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. We need to solve the system $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}$, $\boldsymbol{y}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. It is easy to check that

$$e^{At} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} e^{-2t}.$$

It follows that

$$\boldsymbol{y}_h(t) = \left(\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} e^{-2t} \right) \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} e^{-t} + \begin{bmatrix} 4 \\ -8 \end{bmatrix} e^{-2t}$$

and

$$\begin{aligned} \boldsymbol{y}_p &= e^{At} * \begin{bmatrix} 2 \\ 0 \end{bmatrix} = (e^{-t} * 1) \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (e^{-2t} * 1) \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= (1 - e^{-t}) \begin{bmatrix} 0 \\ 4 \end{bmatrix} + (1 - e^{-2t}) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -4 \end{bmatrix} e^{-t} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}. \end{aligned}$$

It now follows that

$$\begin{aligned} \boldsymbol{y}(t) &= \boldsymbol{y}_h(t) + \boldsymbol{y}_p(t) \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 3 \\ -6 \end{bmatrix} e^{-2t}. \end{aligned}$$

The concentration of salt in Tank 2 is 1/2 if $y_2(t) = 1$. We thus solve $y_2(t) = 1$, i.e. $2 + 4e^{-t} - 6e^{-2t} = 1$ for t. Let $x = e^{-t}$. Then $-6x^2 + 4x + 1 = 0$. The quadratic formula gives $x = \frac{2 \pm \sqrt{10}}{6}$. Since $x > 0$ we have $e^{-t} = x = \frac{2+\sqrt{10}}{6}$. Solving for t we get $t = -\ln\left(\frac{2+\sqrt{10}}{6}\right) = 0.1504$ minutes or 9.02 seconds.

21. y_1 and y_2 are related to each other as follows:

$$\begin{aligned} y_1' &= 2 + 3y_2 - 5y_1 \\ y_2' &= 2 + 5y_1 - 7y_2 \end{aligned}$$

with initial conditions $y_1(0) = 0$ and $y_2(0) = 0$. Let $A = \begin{bmatrix} -5 & 3 \\ 5 & -7 \end{bmatrix}$, $\boldsymbol{f}(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We need to solve the system $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}$, $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. It is easy to check that

$$e^{At} = \frac{e^{-2t}}{8} \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} + \frac{e^{-10t}}{8} \begin{bmatrix} 3 & -3 \\ -5 & 5 \end{bmatrix}.$$

Clearly $\boldsymbol{y}_h = 0$ while

$$\begin{aligned} \boldsymbol{y}(t) = \boldsymbol{y}_p(t) &= e^{At} * \boldsymbol{f}(t) \\ &= \left(\frac{e^{-2t}}{8} \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} + \frac{e^{-10t}}{8} \begin{bmatrix} 3 & -3 \\ -5 & 5 \end{bmatrix} \right) * 1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \frac{e^{-2t} * 1}{8} \begin{bmatrix} 16 \\ 16 \end{bmatrix} \\ &= (1 - e^{-2t}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} y_1(t) &= 1 - e^{-2t} \\ y_2(t) &= 1 - e^{-2t}. \end{aligned}$$

Section 9.6

1. The characteristic polynomial is $c_A(s) = s^2 - 1 = (s+1)(s-1)$. There are two distinct eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = 1$. An easy calculation give that $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue -1 and $v_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 1. Let $P = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$. Then $J = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Since there is a distinct positive and negative real eigenvalue the critical point is a saddle.

3. The characteristic polynomial is $c_A(s) = s^2 + 4s + 5 = (s+2)^2 + 1$ and has complex roots $-2 \pm i$. A calculation gives an eigenvector $v = \begin{bmatrix} -3 - i \\ 5 \end{bmatrix}$ for

$-2-i$. Let $v_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Let $P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & 0 \end{bmatrix}$, Then $J = P^{-1}AP = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$ and the origin is a stable spiral node.

5. In this case A is of type J_3 with positive eigenvalue 2. The origin is an unstable star node.

7. The characteristic polynomial is $c_A(s) = s^2-6s+8 = (s-2)(s-4)$. There are two distinct eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 4$. An easy calculation gives that $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 2 and $v_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 4. Let $P = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$. Then $J = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$. Since both eigenvalues are positive the origin is an unstable node.

9. The characteristic polynomial is $c_A(s) = s^2 - 2s + 5 = (s-1)^2 + 2^2$. So $1 \pm 2i$ are the eigenvalues. An eigenvector for $1-2i$ is $\begin{bmatrix} -1+i \\ 4 \end{bmatrix}$. Let $P = \begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}$. Then $J = P^{-1}AP = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The origin is an unstable star node.

11. Let (x, y) be a point on the $P(L)$. Suppose $P^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $\begin{bmatrix} u \\ v \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$. Then (u, v) is on L and so

$$\begin{aligned} 0 &= Du + Ev + F \\ &= D(ax+by) + E(cx+dy) + F \\ &= (Da+Ec)x + (Db+Ed)y + F \\ &= D'x + E'y + F, \end{aligned}$$

where $(D', E') = (Da+Ec, Db+Ed) = (D, E)P^{-1}$. It follows that (x, y) satisfies the equation of a line. A line goes through the origin if and only if $F = 0$. If the equation for L has $F = 0$ then the above calculation shows the equation for $P(L)$ does too.

13. Let C be the graph of a power curve in the (u, v) plane and $P(C)$ the transform of C. Let (x, y) be a point of $P(C)$ and (u, v) the point on C such that $P \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. If $P^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$. Replace u and v in the equation $Au + Bv = (Cu + Dv)^p$ by $ax + by$ and $cx + dy$, respectively. We then get $(Aa + Bc)x + (Ab + Bd)y = ((Ca+Dc)x + (Cb+Dd)y)^p$. Thus $P(C)$ is the graph of a power curve.

15. The characteristic polynomial takes the form $c_A(s) = s^2 - (\operatorname{tr} A)s + \det A$. Let $\lambda = \operatorname{tr} A$. Since $\det A = 0$ we have $c_A(s) = s^2 - \lambda s = s(s - \lambda)$. Now consider two cases:

$\lambda \neq 0$: In this case A has two distinct eigenvalues, 0 and λ. Let v_1 be an eigenvector with eigenvalue 0 and v_2 an eigenvector with eigenvalue λ. Then v_1 and v_2 are linearly independent. If $P = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ then P is invertible and

$$AP = \begin{bmatrix} Av_1 & Av_2 \end{bmatrix} = \begin{bmatrix} 0 & \lambda v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = PJ_1.$$

Now multiply both sides on the left by P^{-1} to get that $P^{-1}AP = J_1$.

$\lambda = 0$: In this case $c_A(s) = s^2$. Since A is not zero there must be a vector v_1 that is not an eigenvector. Let $v_2 = Av_1$. Then v_2 is an eigenvector with eigenvalue 0 since, by the Cayley-Hamilton theorem,

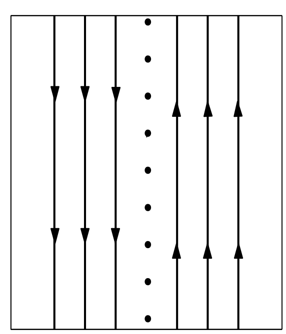
$$Av_2 = A^2 v_1 = 0.$$

Now let $P = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$. Then

$$AP = A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} Av_1 & Av_2 \end{bmatrix} = \begin{bmatrix} v_2 & 0 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = P \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Now multiply both sides on the left by P^{-1} to get that $P^{-1}AP = J_2$.

17. If $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ then the equation $J_2 \boldsymbol{c} = 0$ implies $c_1 = 0$. It follows that each point on the v-axis is an equilibrium point. Now assume $c_1 \neq 0$. The solution to $\boldsymbol{w}' = J_2 \boldsymbol{w}$, $\boldsymbol{w}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is $u(t) = c_1$ and $v(t) = tc_1 + c_2$. The path $(u(t), v(t)) = (c_1, c_2) + t(0, c_1)$, $t \in \mathbb{R}$, is a vertical line that passes through the initial condition (c_1, c_2) and points upward if $c_1 > 0$ and downward it $c_1 < 0$. The phase portrait is given below:



19. It is not difficult to see that $e^{At} = I + tA$. Let v_1 be an eigenvector. By Lemma 9.5.9, $e^{At}v_1 = v_1$. So each eigenvector is an equilibrium point. Let v be a vector that is not an eigenvector. By the Cayley-Hamilton theorem $A^2v = 0$ so Av is an eigenvector. Furthermore $e^{At}v = v + tAv$. The trajectory is a line parallel to Av going through v.

21. Assume $c_1 > 0$ and thus $x > 0$ (the case where $c_1 < 0$ is similar). We have $y' = \dfrac{\ln x/c_1}{\lambda} + \dfrac{1}{\lambda} + \dfrac{c_2}{c_1}$. It follows that $\lim_{x\to 0^+} y' = -\infty$ if $\lambda > 0$ and $\lim_{x\to 0^+} y' = \infty$ if $\lambda < 0$.

23. $y'' = \dfrac{1}{\lambda x}$ and the result follows.

Section 9.7

1. Observe that $\boldsymbol{\Phi}'(t) = \begin{bmatrix} -e^{-t} & 2e^{2t} \\ -e^{-t} & 8e^{2t} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} e^{-t} & e^{2t} \\ e^{-t} & 4e^{2t} \end{bmatrix} = A(t)\boldsymbol{\Phi}(t)$. Also, $\det \boldsymbol{\Phi}(t) = 4e^t - e^t = 3e^t \neq 0$. Thus $\boldsymbol{\Phi}(t)$ is a fundamental matrix. The general solution can be written in the form $\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}$, where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a constant vector. The initial condition implies $\boldsymbol{y}(0) = \boldsymbol{\Phi}(0)\boldsymbol{c}$ or

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 4c_2 \end{bmatrix}.$$

Solving for $\boldsymbol{c}$ we get $c_1 = 2$ and $c_2 = -1$. It follows that

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c} = 2\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} - \begin{bmatrix} e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{2t} \\ 2e^{-t} - 4e^{2t} \end{bmatrix}.$$

The standard fundamental matrix at $t = 0$ is

$$\begin{aligned} \boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(0)^{-1} &= \begin{bmatrix} e^{-t} & e^{2t} \\ e^{-t} & 4e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^{-1} \\ &= \frac{1}{3} \begin{bmatrix} e^{-t} & e^{2t} \\ e^{-t} & 4e^{2t} \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 4e^{-t} - e^{2t} & -e^{-t} + e^{2t} \\ 4e^{-t} - 4e^{2t} & -e^{-t} + 4e^{2t} \end{bmatrix}. \end{aligned}$$

3. Observe that

$$\begin{aligned}\boldsymbol{\Phi}'(t) &= \begin{bmatrix} t\cos(t^2/2) & -t\sin(t^2/2) \\ -t\sin(t^2/2) & -\cos(t^2/2)\end{bmatrix} \\ &= \begin{bmatrix} 0 & t \\ -t & 0\end{bmatrix}\begin{bmatrix}\sin(t^2/2) & \cos(t^2/2) \\ \cos(t^2/2) & -\sin(t^2/2)\end{bmatrix} \\ &= A(t)\boldsymbol{\Phi}(t).\end{aligned}$$

Also, $\det \boldsymbol{\Phi}(t) = -\sin^2(t^2/2) - \cos^2(t^2/2) = -1 \neq 0$. Thus $\boldsymbol{\Phi}(t)$ is a fundamental matrix. The general solution can be written in the form $\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}$, where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a constant vector. The initial condition implies $\boldsymbol{y}(0) = \boldsymbol{\Phi}(0)\boldsymbol{c}$ or

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}.$$

Thus $c_1 = 0$ and $c_2 = 1$. It follows that

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c} = 0\begin{bmatrix}\sin(t^2/2) \\ \cos(t^2/2)\end{bmatrix} + \begin{bmatrix}\cos(t^2/2) \\ -\sin(t^2/2)\end{bmatrix} = \begin{bmatrix}\cos(t^2/2) \\ -\sin(t^2/2)\end{bmatrix}.$$

The standard fundamental matrix at $t = 0$ is

$$\begin{aligned}\boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(0)^{-1} &= \begin{bmatrix}\sin(t^2/2) & \cos(t^2/2) \\ \cos(t^2/2) & -\sin(t^2/2)\end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}^{-1} \\ &= \begin{bmatrix}\sin(t^2/2) & \cos(t^2/2) \\ \cos(t^2/2) & -\sin(t^2/2)\end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix} \\ &= \begin{bmatrix}\cos(t^2/2) & \sin(t^2/2) \\ -\sin(t^2/2) & \cos(t^2/2)\end{bmatrix}.\end{aligned}$$

5. Observe that

$$\begin{aligned}\boldsymbol{\Phi}'(t) &= \begin{bmatrix} -\cos t + t\sin t & -\sin t - t\cos t \\ \sin t + t\cos t & -\cos t + t\sin t\end{bmatrix} \\ &= \begin{bmatrix} 1/t & 1 \\ -1 & 1/t\end{bmatrix}\begin{bmatrix} -t\cos t & -t\sin t \\ t\sin t & -t\cos t\end{bmatrix} \\ &= A(t)\boldsymbol{\Phi}(t).\end{aligned}$$

Also, $\det \boldsymbol{\Phi}(t) = t^2\cos^2 t + t^2\sin^2 t = t^2 \neq 0$. Thus $\boldsymbol{\Phi}(t)$ is a fundamental matrix. The general solution can be written in the form $\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}$, where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a constant vector. The initial condition implies $\boldsymbol{y}(\pi) = \boldsymbol{\Phi}(\pi)\boldsymbol{c}$ or

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \pi c_1 \\ \pi c_2 \end{bmatrix}.$$

Thus $c_1 = 1/\pi$ and $c_2 = -1/\pi$. It follows that

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c} = \frac{1}{\pi}\begin{bmatrix} -t\cos t \\ t\sin t \end{bmatrix} - \frac{1}{\pi}\begin{bmatrix} -t\sin t \\ -t\cos t \end{bmatrix} = \frac{t}{\pi}\begin{bmatrix} -\cos t + \sin t \\ \cos t + \sin t \end{bmatrix}.$$

The standard fundamental matrix at $t = \pi$ is

$$\begin{aligned} \boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(\pi)^{-1} &= \begin{bmatrix} -t\cos t & -t\sin t \\ t\sin t & -t\cos t \end{bmatrix}\begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}^{-1} \\ &= \frac{1}{\pi}\begin{bmatrix} -t\cos t & -t\sin t \\ t\sin t & -t\cos t \end{bmatrix}. \end{aligned}$$

7. Observe that

$$\begin{aligned} \boldsymbol{\Phi}'(t) &= \begin{bmatrix} 0 & te^t \\ 0 & e^t \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1/t & 1/t \end{bmatrix}\begin{bmatrix} 1 & (t-1)e^t \\ -1 & e^t \end{bmatrix} \\ &= A(t)\boldsymbol{\Phi}(t). \end{aligned}$$

Also, $\det \boldsymbol{\Phi}(t) = e^t + te^t - e^t = te^t \neq 0$. Thus $\boldsymbol{\Phi}(t)$ is a fundamental matrix. The general solution can be written in the form $\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}$, where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a constant vector. The initial condition implies $\boldsymbol{y}(0) = \boldsymbol{\Phi}(0)\boldsymbol{c}$ or

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & e \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_1 + ec_2 \end{bmatrix}.$$

Thus $c_1 = -3$ and $c_2 = 1/e$. It follows that

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c} = -3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{e}\begin{bmatrix} (t-1)e^t \\ e^t \end{bmatrix} = \begin{bmatrix} (t-1)e^{t-1} - 3 \\ e^{t-1} + 3 \end{bmatrix}.$$

The standard fundamental matrix at $t = 0$ is

$$\begin{aligned} \boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(0)^{-1} &= \begin{bmatrix} 1 & (t-1)e^t \\ -1 & e^t \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & e \end{bmatrix}^{-1} \\ &= \frac{1}{e}\begin{bmatrix} 1 & (t-1)e^t \\ -1 & e^t \end{bmatrix}\begin{bmatrix} e & 0 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{e}\begin{bmatrix} e + (t-1)e^t & (t-1)e^t \\ -e + e^t & e^t \end{bmatrix} \\ &= \begin{bmatrix} 1 + (t-1)e^{t-1} & (t-1)e^{t-1} \\ -1 + e^{t-1} & e^{t-1} \end{bmatrix}. \end{aligned}$$

9. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$. Then $A(t) = \frac{1}{t}A$. We first compute e^{Au}. The characteristic polynomial, $c_A(s)$ is

$$c_A(s) = \det(sI - A) = \det \begin{bmatrix} s & 1 \\ -1 & s-2 \end{bmatrix} = s^2 - 2s + 1 = (s-1)^2.$$

It follows that $\mathcal{B}_{c_A} = \{e^u, ue^u\}$. Using Fulmer's method we have $e^{Au} = e^u M_1 + ue^u M_2$. Differentiating and evaluating at $u = 0$ gives

$$\begin{aligned} I &= M_1 \\ A &= M_1 + M_2. \end{aligned}$$

It follows that $M_1 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $M_2 = A - I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. Thus

$$e^{Au} = e^u \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ue^u \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Since $\ln t$ is an antiderivative of $\frac{1}{t}$ and $\ln 1 = 0$ we have by Proposition 12

$$\begin{aligned} \boldsymbol{\Psi}(t) &= e^{\ln t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\ln t)e^{\ln t} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} t - t\ln t & -t\ln t \\ t\ln t & t + t\ln t \end{bmatrix}, \end{aligned}$$

is the standard fundamental matrix for $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t)$ at $t = 1$. The homogeneous solution is given by

$$\begin{aligned} \boldsymbol{y}_h(t) &= \boldsymbol{\Psi}(t)\boldsymbol{y}(1) \\ &= \begin{bmatrix} t - t\ln t & -t\ln t \\ t\ln t & t + t\ln t \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2t - 2t\ln t \\ 2t\ln t \end{bmatrix}. \end{aligned}$$

The particular solution is given by

$$\begin{aligned} \boldsymbol{y}_p(t) &= \boldsymbol{\Psi}(t) \int_1^t \boldsymbol{\Psi}(u)^{-1} \boldsymbol{f}(u)\, du \\ &= \begin{bmatrix} t - t\ln t & -t\ln t \\ t\ln t & t + t\ln t \end{bmatrix} \int_1^t \frac{1}{u} \begin{bmatrix} 1 + \ln u & \ln u \\ -\ln u & 1 - \ln u \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} du \\ &= \begin{bmatrix} t - t\ln t & -t\ln t \\ t\ln t & t + t\ln t \end{bmatrix} \ln t \begin{bmatrix} 1 \\ -1 \end{bmatrix} du \\ &= t\ln t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

It follows that

$$\boldsymbol{y}(t) = \boldsymbol{y}_h(t) + \boldsymbol{y}_p(t) = \begin{bmatrix} 2t - 2t\ln t \\ 2t\ln t \end{bmatrix} + \begin{bmatrix} t\ln t \\ -t\ln t \end{bmatrix} = \begin{bmatrix} 2t - t\ln t \\ t\ln t \end{bmatrix}.$$

11. Let $A = \begin{bmatrix} 3 & 5 \\ -1 & -3 \end{bmatrix}$. Then $A(t) = \sec(t)A$. The characteristic polynomial of A is $c_A(s) = s^2 - 4 = (s-2)(s+2)$. Hence $\mathcal{B}_{c_A} = \left\{e^{2t}, e^{-2t}\right\}$ and

$$e^{Au} = M_1 e^{2u} + M_2 e^{-2u}.$$

Differentiating and evaluating at $u = 0$ give the equations $I = M_1 + M_2$ and $A = 2M_1 - 2M_2$. It follows that

$$M_1 = \frac{1}{4}\begin{bmatrix} 5 & 5 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad M_2 = \frac{1}{4}\begin{bmatrix} -1 & -5 \\ 1 & 5 \end{bmatrix}$$

and

$$e^{Au} = \frac{1}{4}\begin{bmatrix} 5e^{2u} - e^{-2u} & 5e^{2u} - 5e^{-2u} \\ -e^{2u} + e^{-2u} & -e^{2u} + 5e^{-2u} \end{bmatrix}.$$

If $b(t) = \int_0^t \sec u \, du = \ln|\sec t + \tan t|$ then $\boldsymbol{\Psi}(t) = e^{Ab(t)}$. If $X = (\sec t + \tan t)^2$ then $X^{-1} = (\sec t - \tan t)^2$ and

$$\begin{aligned} \boldsymbol{\Psi}(t) &= \frac{1}{4}\begin{bmatrix} 5X - \frac{1}{X} & 5X - 5\frac{1}{X} \\ -X + \frac{1}{X} & -X + 5\frac{1}{X} \end{bmatrix} \\ &= \begin{bmatrix} \sec^2 t + 3\sec t\tan t + \tan^2 t & 5\sec t\tan t \\ -\sec t\tan t & \sec^2 t - 3\sec t\tan t + \tan^2 t \end{bmatrix}. \end{aligned}$$

From this it follows that the homogeneous solution is

$$\begin{aligned} \boldsymbol{y}_h(t) &+ \begin{bmatrix} \sec^2 t + 3\sec t\tan t + \tan^2 t & 5\sec t\tan t \\ -\sec t\tan t & \sec^2 t - 3\sec t\tan t + \tan^2 t \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2\sec^2 t + 11\sec t\tan t + 2\tan^2 t \\ \sec^2 t - 5\sec t\tan t + \tan^2 t \end{bmatrix} \end{aligned}$$

Since the forcing function $\boldsymbol{f}$ is identically zero the particular solution is zero. Hence $\boldsymbol{y} = \boldsymbol{y}_h$.

13. Let $v_1(t)$ and $v_2(t)$ denote the volume of brine in Tank 1 and Tank 2, respectively. Then $v_1(t) = v_2(t) = 2 - t$. The following differential equations describe the system

$$\begin{aligned} y_1'(t) &= -\frac{3}{2-t}y_1(t) + \frac{1}{2-t}y_2(t) + 6 \\ y_2'(t) &= \frac{1}{2-t}y_1(t) - \frac{3}{2-t}y_2(t) + 0, \end{aligned}$$

with initial conditions $y_1(0) = 0$ and $y_2(0) = 20$. In matrix form, $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t) + \boldsymbol{f}(t)$, we have

$$A(t) = \begin{bmatrix} -3/(2-t) & 1/(2-t) \\ 1/(2-t) & -3/(2-t) \end{bmatrix}, \ \boldsymbol{f}(t) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \ \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 20 \end{bmatrix}.$$

Let $a(t) = \dfrac{-1}{2-t}$. Then we can write $A(t) = a(t)A$ where

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

The characteristic polynomial is $c_A(s) = s^2 - 6s + 8 = (s-2)(s-4)$. We now have $\mathcal{B}_{c_A} = \left\{e^{2t}, e^{4t}\right\}$ It follows that $e^{Au} = M_1 e^{2u} + M_2 e^{4u}$. Differentiating and setting $u = 0$ we get

$$\begin{aligned} I &= M_1 + M_2 \\ A &= 2M_1 + 4M_2. \end{aligned}$$

An easy calculation gives

$$\begin{aligned} M_1 &= \frac{1}{2}(4I - A) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ M_2 &= \frac{1}{2}(A - 2I) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

and

$$e^{Au} = \frac{e^{2u}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{e^{4u}}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Let $b(t) = \int_0^t a(u)\,du = \int_0^t \dfrac{-1}{2-u}\,du = \ln \frac{2-t}{2}$. Then the standard fundamental matrix is

$$\boldsymbol{\Psi}(t) = e^{Au}|_{u=b(t)} = \frac{(2-t)^2}{8}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{(2-t)^4}{32}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

For the homogeneous solution we have

$$\begin{aligned}
\boldsymbol{y}_h(t) &= \boldsymbol{\Psi}(t)\boldsymbol{y}(0) \\
&= \frac{(2-t)^2}{8}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ 20\end{bmatrix} + \frac{(2-t)^4}{32}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}0\\ 20\end{bmatrix} \\
&= \frac{(2-t)^2}{2}\begin{bmatrix}5\\ 5\end{bmatrix} + \frac{(2-t)^4}{8}\begin{bmatrix}-5\\ 5\end{bmatrix}.
\end{aligned}$$

For the particular solution straightforward calculations give

$$\boldsymbol{\Psi}^{-1}(u) = \frac{2}{(2-u)^4}\begin{bmatrix}(2-u)^2+4 & (2-u)^2-4\\ (2-u)^2-4 & (2-u)^2+4\end{bmatrix},$$

$$\begin{aligned}
\boldsymbol{\Psi}^{-1}\boldsymbol{f}(u) &= \frac{2}{(2-u)^4}\begin{bmatrix}(2-u)^2+4 & (2-u)^2-4\\ (2-u)^2-4 & (2-u)^2+4\end{bmatrix}\begin{bmatrix}6\\ 0\end{bmatrix} \\
&= \frac{12}{(2-u)^4}\begin{bmatrix}(2-u)^2+4\\ (2-u)^2-4\end{bmatrix} \\
&= 12\begin{bmatrix}(2-u)^{-2}+4(2-u)^{-4}\\ (2-u)^{-2}-4(2-u)^{-4}\end{bmatrix},
\end{aligned}$$

and

$$\int_0^t \boldsymbol{\Psi}^{-1}(u)\boldsymbol{f}(u)\,du = \frac{4}{(2-t)^3}\begin{bmatrix}3(2-t)^2+4\\ 3(2-t)^2-4\end{bmatrix} - \begin{bmatrix}8\\ 4\end{bmatrix}.$$

Finally, we get

$$\begin{aligned}
\boldsymbol{y}_p(t) &= \boldsymbol{\Psi}(t)\int_0^t \boldsymbol{\Psi}^{-1}(u)\boldsymbol{f}(u)\,du \\
&= \left(\frac{(2-t)^2}{8}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \frac{(2-t)^4}{32}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right)\int_0^t \boldsymbol{\Psi}^{-1}(u)\boldsymbol{f}(u)\,du \\
&= (2-t)\begin{bmatrix}4\\ 2\end{bmatrix} - \frac{(2-t)^2}{2}\begin{bmatrix}3\\ 3\end{bmatrix} - \frac{(2-t)^4}{8}\begin{bmatrix}1\\ -1\end{bmatrix}.
\end{aligned}$$

We now add the homogeneous and particular solutions together and simplify to get

$$\begin{aligned}
\boldsymbol{y}(t) &= \boldsymbol{y}_h + \boldsymbol{y}_p \\
&= (2-t)\begin{bmatrix}4\\ 2\end{bmatrix} + (2-t)^2\begin{bmatrix}1\\ 1\end{bmatrix} + \frac{(2-t)^4}{4}\begin{bmatrix}-3\\ 3\end{bmatrix}.
\end{aligned}$$

We now get

$$\begin{aligned} y_1(t) &= 4(2-t) + (2-t)^2 - \frac{3}{4}(2-t)^4 \\ y_2(t) &= 2(2-t) + (2-t)^2 + \frac{3}{4}(2-t)^3. \end{aligned}$$

The amount of fluid in each tank after 1 minute is $v_1(1) = v_2(1) = 1$. Thus the concentrations (grams/L) of salt in Tank 1 is $y_1(1)/1$ and in Tank 2 is $y_2(1)/1$, i.e.

$$\frac{y_1(1)}{1} = \frac{17}{4} \quad \text{and} \quad \frac{y_2(1)}{1} = \frac{15}{4}.$$

Springer

http://www.springer.com/978-1-4614-3617-1

Ordinary Differential Equations
Adkins, W.A.; Davidson, M.G.
2012, XIII, 799 p. 121 illus., Hardcover
ISBN: 978-1-4614-3617-1