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Appendix A

COMPLEX NUMBERS

A.1 Complex Numbers

The history of numbers starts in the stone age, about 30,000 years ago. Long before humans could read or write, a caveman who counted the deer he killed by a series of notches carved into a bone, introduced mankind to the natural counting numbers $1, 2, 3, 4, \cdots$. To be able to describe quantities and their relations among each other, the first human civilizations expanded the number system first to rational numbers (integers and fractions) and then to real numbers (rational numbers and irrational numbers like $\sqrt{2}$ and π). Finally in 1545, to be able to tackle more advanced computational problems in his book about *The Great Art (Ars Magna)*, Girolamo Cardano brought the complex numbers (real numbers and "imaginary" numbers like $\sqrt{-1}$) into existence. Unfortunately, 450 years later and after changing the whole of mathematics forever, complex numbers are still greeted by the general public with suspicion and confusion.

The problem is that most folks still think of numbers as entities that are used solely to describe quantities. This works reasonably well if one restricts the number universe to the real numbers, but fails miserably if one considers complex numbers: no one will ever catch $\sqrt{-1}$ pounds of crawfish, not even a mathematician.

In mathematics, numbers are used to do computations, and it is a matter of fact that nowaday~~s almost~~ all serious computations in mathematics require somewhere along the line the ~~use of the~~ largest possible number system given to mankind: the complex numbers. Although complex numbers are useless to describe the weight of your catch of the day, they are indispensable if, for example, you want to make a sound mathematical prediction about the behavior of any biological, chemical, or physical system in time.

Since the ancient Greeks, the algebraic concept of a real number is associated with the geometric concept of a point on a line (the *number line*), and these two concepts are still used as synonyms. Similarly, complex numbers can be given a simple, concrete, geometric interpretation as points in a plane; i.e., any **complex number** z corresponds to a point in the plane (the *number plane*) and can be represented in Cartesian coordinates as $z = (x, y)$, where x and y are real numbers.

We know from Calculus II that every point $z = (x, y)$ in the plane can be described also in **polar coordinates** as $z = [\alpha, r]$, where $r = |z| = \sqrt{x^2 + y^2}$ denotes the **radius (length, modulus, norm, absolute value, distance to the origin)** of the point z, and where $\alpha = \arg(z)$ is the angle (in radians) between the positive x-axis and the line joining 0 and z. Note that α can be determined by the equation $\tan \alpha = y/x$, when $x \neq 0$, and knowledge of which quadrant the number z is in. Be aware that α is not unique; adding $2\pi k$ to α gives another angle (argument) for z.

We identify the real numbers with the x-axis in the plane; i.e., a real number x is identified with the point $(x, 0)$ of the plane, and vice versa. Thus, the real numbers are a subset of the complex numbers. As pointed out above, in mathematics the defining property of numbers is not that they describe quantities, but that we can do computations with them; i.e., we should be able to add and multiply them. The addition and multiplication of points in the plane are defined in such a way that

(a) they coincide on the x-axis (real numbers) with the usual addition and multiplication of real numbers, and

(b) all rules of algebra for real numbers (points on the x-axis) extend to complex numbers (points in the plane).

Addition: we add complex numbers coordinate-wise in Cartesian coordinates. That is, if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2).$$

Multiplication: we multiply complex numbers in polar coordinates by adding their angles α and multiplying their radii r (in polar coordinates). That is, if $z_1 = [\alpha_1, r_1]$ and $z_2 = [\alpha_2, r_2]$, then

$$z_1 z_2 := [\alpha_1 + \alpha_2, r_1 r_2].$$

The definition of multiplication of points in the plane is an extension of the familiar rule for multiplication of signed real numbers: plus times plus is plus, minus times minus is plus, plus times minus is minus. To see this, we identify the real numbers 2 and −3 with the complex numbers $z_1 = (2, 0) = [0, 2]$ and $z_2 = (-3, 0) = [\pi, 3]$. Then

$$z_1 z_2 = [0 + \pi, 2 \cdot 3] = [\pi, 6] = (-6, 0) = -6$$
$$z_2^2 = [\pi, 3][\pi, 3] = [\pi + \pi, 3 \cdot 3] = [2\pi, 9] = (0, 9) = 9,$$

which is not at all surprising since we all know that $2 \cdot -3 = -6$, and $(-3)^2 = 9$. What this illustrates is part (a); namely, the arithmetic of real numbers is the same whether considered in their own right, or considered as a subset of the complex numbers.

To demonstrate the multiplication of complex numbers (points in the plane) which are not real (not on the x-axis), consider $z_1 = (1, 1) = [\frac{\pi}{4}, \sqrt{2}]$ and $z_2 = (1, -1) = [-\frac{\pi}{4}, \sqrt{2}]$. Then

$$z_1 z_2 = [\frac{\pi}{4} - \frac{\pi}{4}, \sqrt{2} \cdot \sqrt{2}] = [0, 2] = (2, 0) = 2.$$

If one defines multiplication of points in the plane as above, the point $i := (0, 1) = [\frac{\pi}{2}, 1]$ has the property that

$$i^2 = [\frac{\pi}{2} + \frac{\pi}{2}, 1 \cdot 1] = [\pi, 1] = (-1, 0) = -1.$$

Thus, one defines

$$\sqrt{-1} := i = (0, 1).$$

Notice that $\sqrt{-1}$ is not on the x-axis and is therefore not a real number. Employing i and identifying the point $(1, 0)$ with the real number 1, one can now write a complex number $z = (x, y)$ in the **standard algebraic form** $z = x + iy$; i.e.,

$$z = (x, y) = (x, 0) + (0, y) = x(1, 0) + (0, 1)y = x + iy.$$

If $z = (x, y) = x + iy$, then the real number $x := \operatorname{Re} z$ is called the **real part** and the real number $y := \operatorname{Im} z$ is called the **imaginary part** of z (which is one of the worst misnomers in the history of science since there is absolutely nothing imaginary about y).

The basic rules of algebra carry over to complex numbers if we simply remember the identity $i^2 = -1$. In particular, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + iy_1x_2 + x_1iy_2 + iy_1iy_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \end{aligned}$$

This algebraic rule is often easier to use than the geometric definition of multiplication given above. For example, if $z_1 = (1, 1) = 1 + i$ and $z_2 = (1, -1) = 1 - i$, then the computation $z_1 z_2 = (1+i)(1-i) = 1 - i^2 = 2$ is more familiar than the one given above using the polar coordinates of z_1 and z_2.

The formula for **division** of two complex numbers (points in the plane) is less obvious, and is most conveniently expressed in terms of the **complex conjugate** $\overline{z} := (x, -y) = x - iy$ of a complex number $z = (x, y) = x + iy$. Note that $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$, and

$$|z|^2 = x^2 + y^2 = z\overline{z}, \qquad \operatorname{Re} z = \frac{z + \overline{z}}{2} \quad \text{and} \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$

Using complex conjugates, we divide complex numbers using the formula

$$\frac{z}{w} = \frac{z}{w} \cdot \frac{\overline{w}}{\overline{w}} = \frac{z\overline{w}}{|w|^2}.$$

As an example we divide the complex number $z = (1, 1) = 1 + i$ by $w = (3, -1) = 3 - i$. Then

$$\frac{z}{w} = \frac{1+i}{3-i} = \frac{(1+i)(3+i)}{(3-i)(3+i)} = \frac{2+4i}{10} = \frac{1}{5} + \frac{2}{5}i = (\frac{1}{5}, \frac{2}{5}).$$

Let $z = (x, y)$ be a complex number with polar coordinates $z = [\alpha, r]$. Then $|z| = r = \sqrt{x^2 + y^2}$, $\operatorname{Re} z = x = |z| \cos \alpha$, $\operatorname{Im} z = y = |z| \sin \alpha$, and $\tan \alpha = y/x$. Thus we obtain the following **exponential form** of the complex number z; i.e.,

$$z = [\alpha, r] = (x, y) = |z|(\cos \alpha, \sin \alpha) = |z|(\cos \alpha + i \sin \alpha) = |z|e^{i\alpha},$$

where the last identity requires Euler's formula relating the complex exponential and trigonometric functions. The most natural means of understanding the validity of Euler's formula is via the power series expansions of e^x, $\sin x$, and $\cos x$, which were studied in calculus. Recall that the exponential function e^x has a power series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

which converges for all $x \in \mathbb{R}$. This infinite series makes perfectly good sense if x is replaced by *any* complex number z, and moreover, it can be shown that the resulting series converges for all $z \in \mathbb{C}$. Thus, we *define* the **complex exponential function** by means of the convergent series

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{1}$$

It can be shown that this function e^z satisfies the expected functional equation, that is

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}.$$

Since $e^0 = 1$, it follows that $\frac{1}{e^z} = e^{-z}$. Euler's formula will be obtained by taking $z = it$ in Definition 1; i.e.,

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \cdots \\ &= (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots) + i(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots) = \cos t + i \sin t = (\cos t, \sin t), \end{aligned}$$

where one has to know that the two series following the last equality are the Taylor series expansions for $\cos t$ and $\sin t$, respectively. Thus we have proved Euler's formula, which we formally state as a theorem.

Theorem A.1.1 (Euler's Formula). *For all $t \in \mathbb{R}$ we have*

$$e^{it} = \cos t + i \sin t = (\cos t, \sin t) = [t, 1].$$ □

Example A.1.2. Write $z = -1 + i$ in exponential form.

▶ **Solution.** Note that $z = (-1, 1)$ so that $x = -1$, $y = 1$, $r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, and $\tan \alpha = y/x = -1$. Thus, $\alpha = \frac{3\pi}{4}$ or $\alpha = \frac{7\pi}{4}$. But z is in the 2nd quadrant, so $\alpha = \frac{3\pi}{4}$. Thus the polar coordinates of z are $[\frac{3\pi}{4}, \sqrt{2}]$ and the exponential form of z is $\sqrt{2}e^{i\frac{3\pi}{4}}$. ◀

Example A.1.3. Write $z = 2e^{\frac{\pi i}{6}}$ in Cartesian form.

▶ **Solution.**

$$z = 2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i = (\sqrt{3}, 1).$$

◀

Using the exponential form of a complex number gives yet another description of the multiplication of two complex numbers. Suppose that z_1 and z_2 are given in exponential form, that is, $z_1 = r_1e^{i\alpha_1}$ and $z_2 = r_2e^{i\alpha_2}$. Then

$$z_1z_2 = (r_1e^{i\alpha_1})(r_2e^{i\alpha_2}) = (r_1r_2)e^{i(\alpha_1+\alpha_2)}.$$

Of course, this is nothing more than a reiteration of the definition of multiplication of complex numbers; i.e., if $z_1 = [\alpha_1, r_1]$ and $z_2 = [\alpha_2, r_2]$, then $z_1z_2 := [\alpha_1 + \alpha_2, r_1r_2]$.

Example A.1.4. Find $z = \sqrt{i}$. That is, find all z such that $z^2 = i$.

▶ **Solution.** Observe that $i = (0,1) = [\pi/2, 1] = e^{\frac{\pi}{2}i}$. Hence, if $z = e^{i\frac{\pi}{4}}$ then $z^2 = (e^{i\frac{\pi}{4}})^2 = e^{i\frac{\pi}{2}} = i$ so that

$$z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(1+i).$$

Also note that $i = e^{(\frac{\pi}{2}+2\pi)i}$ so that $w = e^{(\frac{\pi}{4}+\pi)i} = e^{\frac{\pi}{4}}e^{\pi i} = -e^{\frac{\pi}{4}} = -z$ is another square root of i. ◀

Example A.1.5. Find all complex solutions to the equation $z^3 = 1$.

▶ **Solution.** Note that $1 = e^{2\pi ki}$ for any integer k. Thus the cube roots of 1 are obtained by dividing the possible arguments of 1 by 3 since raising a complex number to the third power multiplies the argument by 3 (and also cubes the modulus). Thus the possible cube roots of 1 are 1, $\omega = e^{\frac{2\pi}{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = e^{\frac{4\pi}{3}i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}$. ◀

We will conclude this section by summarizing some of the properties of the complex exponential function. The proofs are straight forward calculations based on Euler's formula and are left to the reader.

Theorem A.1.6. *Let $z = x + iy$. Then*

1. $e^z = e^{x+iy} = e^x\cos y + ie^x\sin y$. *That is* $\operatorname{Re} e^z = e^x\cos y$ *and* $\operatorname{Im} e^z = e^x\sin y$.
2. $|e^z| = e^x$. *That is, the modulus of e^z is the exponential of the real part of z.*
3. $\cos y = \dfrac{e^{iy} + e^{-iy}}{2}$
4. $\sin y = \dfrac{e^{iy} - e^{-iy}}{2i}$ □

Example A.1.7. Compute the real and imaginary parts of the complex function

$$z(t) = (2+3i)e^{i\frac{5t}{2}}.$$

▶ **Solution.** Since $z(t) = (2+3i)(\cos\frac{5t}{2} + i\sin\frac{5t}{2}) = (2\cos\frac{5t}{2} - 3\sin\frac{5t}{2}) + (3\cos\frac{5t}{2} + 2\sin\frac{5t}{2})i$, it follows that $\operatorname{Re} z(t) = 2\cos\frac{5t}{2} - 3\sin\frac{5t}{2}$ and $\operatorname{Im} z(t) = 3\cos\frac{5t}{2} + 2\sin\frac{5t}{2}$. ◀

Exercises

1. Let $z = (1, 1)$ and $w = (-1, 1)$. Find $z \cdot w$, $\frac{z}{w}$, $\frac{w}{z}$, z^2, $\sqrt{z}$ and z^{11} using
   (a) the polar coordinates,
   (b) the standard forms $x + iy$,
   (c) the exponential forms.

2. Find
   (a) $(1 + 2i)(3 + 4i)$ (b) $(1 + 2i)^2$ (c) $\dfrac{1}{2 + 3i}$ (d) $\dfrac{1}{(2 - 3i)(2 + 4i)}$ (e) $\dfrac{4 - 2i}{2 + i}$.

3. Solve each of the following equations for z and check your result.
   (a) $(2 + 3i)z + 2 = i$ (b) $\dfrac{z - 1}{z - i} = \dfrac{2}{3}$ (c) $\dfrac{2 + i}{z} + 1 = 2 + i$ (d) $e^z = -1$.

4. Find the modulus of each of the following complex numbers.
   (a) $4 + 3i$ (b) $(2 + i)^2$ (c) $\dfrac{13}{5 + 12i}$ (d) $\dfrac{1 + 2it - t^2}{1 + t^2}$ where $t \in \mathbb{R}$.

5. Find all complex numbers z such that $|z - 1| = |z - 2|$. What does this equation mean geometrically?

6. Determine the region in the complex plane $\mathbb{C}$ described by the inequality

$$|z - 1| + |z - 3| < 4.$$

   Give a geometric description of the region.

7. Compute: (a) $\sqrt{2 + 2i}$ (b) $\sqrt{3 + 4i}$

8. Write each of the following complex numbers in exponential form.
   (a) $3 + 4i$ (b) $3 - 4i$ (c) $(3 + 4i)^2$ (d) $\dfrac{1}{3 + 4i}$ (e) -5 (f) $3i$

9. Find the real and imaginary parts of each of the following functions.
   (a) $(2 + 3i)e^{(-1+i)t}$ (b) $ie^{2it+\pi}$ (c) $e^{(2+3i)t}e^{(-3-i)t}$

10. (a) Find the value of the sum

$$1 + e^z + e^{2z} + \cdots + e^{(n-1)z}.$$

   Hint: Compare the sum to a finite geometric series.

(b) Compute $\sin(\frac{2\pi}{n}) + \sin(\frac{4\pi}{n}) + \cdots + \sin(\frac{(n-1)\pi}{n})$.

11. Find all of the cube roots of $8i$. That is, find all solutions to the equation $z^3 = 8i$.

12. By multiplying out $e^{i\theta}e^{i\phi}$ and comparing it to $e^{i(\theta+\phi)}$, rederive the addition formulas for the cosine and sine functions.

Appendix A

Section A.1

1. $z = (1,1) = 1 + i = [\frac{\pi}{4}, \sqrt{2}] = \sqrt{2}e^{i\frac{\pi}{4}}$, $w = (-1,1) = -1 + i = [\frac{3\pi}{4}, \sqrt{2}] = \sqrt{2}e^{i\frac{3\pi}{4}}$, $z \cdot w = -2$, $\frac{z}{w} = -i$, $\frac{w}{z} = i$, $z^2 = 2i$, $\sqrt{z} = \pm(\sqrt{\sqrt{2}}\cos(\frac{\pi}{8}) + i\sqrt{\sqrt{2}}\sin(\frac{\pi}{8})) = \pm(\frac{1}{2}\sqrt{2\sqrt{2}+2} + i\frac{1}{2}\sqrt{2\sqrt{2}-2})$ since $\cos(\frac{\pi}{8}) = \frac{1}{2}\sqrt{2+\sqrt{2}}$ and $\sin(\frac{\pi}{8}) = \frac{1}{2}\sqrt{2-\sqrt{2}}$, $z^{11} = -32 + 32i$.

2. (a) $-5 + 10i$ (b) $-3 + 4i$ (c) $\frac{2}{13} - \frac{3}{13}i$ (d) $\frac{8}{130} - \frac{1}{130}i$ (e) $\frac{6}{5} - \frac{8}{5}i$.

3. (a) - (c) check your result (d) $z = (3\pi/2 + 2k\pi)i$ for all integers k.

4. Always either 5 or 1.

5. The vertical line $x = \dfrac{3}{2}$. The distance between two points z, w in the plane is given by $|z - w|$. Hence, the equation describes the set of points z in the plane which are equidistant from 1 and 2.

6. This is the set of points inside the ellipse with foci $(1,0)$ and $(3,0)$ and major axis of length 4.

7. (a) $\pm\left(\sqrt{\sqrt{2}+1} + i\sqrt{\sqrt{2}-1}\right)$ (b) $\pm(2+i)$

8. (a) $5e^{i\tan^{-1}(4/3)} \approx 5e^{0.927i}$ (b) $5e^{-i\tan^{-1}(4/3)}$ (c) $25e^{2i\tan^{-1}(4/3)}$ (d) $\frac{1}{5}e^{-i\tan^{-1}(4/3)}$ (e) $5e^{i\pi}$ (f) $3e^{\frac{i\pi}{2}}$

9. (a) Real: $2e^{-t}\cos t - 3e^{-t}\sin t$; Imaginary: $3e^{-t}\cos t + 2e^{-t}\sin t$ (b) Real: $-e^{\pi}\sin 2t$; Imaginary: $e^{\pi}\cos 2t$ (c) Real: $e^{-t}\cos 2t$; Imaginary: $e^{-t}\sin 2t$

10. (a) If $z = 2\pi ki$ for k an integer, the sum is n. Otherwise the sum is $\dfrac{1-e^{nz}}{1-e^{z}}$. (b) 0

11. $-2i$, $\sqrt{3} + i$, $-\sqrt{3} + 2i$