RESEARCH STATEMENT

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A. Introduction

My research focuses on the structural properties of 3-connected graphs within graph theory. Specifically, I explore extensions and generalizations of classical results, such as Tutte's Wheel Theorem, by extending Tutte's Wheel Theorem and developing new Chain Theorems for specific subclasses of 3-connected graphs. There are many known results regarding the characterization of H-free graphs, where H is a small graph on less than 15 edges. (See [2, 5, 7, 10, 11, 15]). My results help simplify the proof of the earlier known results. Furthermore, this work aims to provide a pathway to developing and improving algorithms that may be used to characterize H-free graphs of edge sizes 15 and above.

In my thesis, I present several new Chain Theorems. I have improved the Tutte's Wheel Theorem by limiting the graph operations to edge addition and a vertex split where one of the new vertex is always cubic. This significantly reduces computational time in algorithms for generating 3-connected graphs. Additionally, I have also generalized the block-tree theorem for a collection of non-crossing separators.

In addition to these theorems, I have implemented Sagemath on Python. The program can construct 3-connected graphs, and perform a minor relation test for a choice of forbidden graph. When the result for a H-free list is finite, the program can generate the list of maximal H-free graphs.

B. Chain Theorems

Below are the Chain Theorems that I have established.

1. Chain Theorem on 3-connected graphs. Tutte's wheel theorem [14] plays a crucial role in characterizing H-free graphs for small H, particularly when H has up to 15 edges. The theorem provides a foundational result in graph theory, describing how wheels are central to understanding 3-connected graphs. For small H, the exclusion of H helps identify important structural properties and leads to characterizations of graphs that avoid certain configurations. This characterization is vital for studying graph minors and contributes to broader areas like graph coloring, planarity, and the decomposition of graphs into simpler components.
We begin with a few definitions.

Definition 1.1. A wheel on $n+1$ vertices ($n \geq 3$), denoted by W_n, is obtained from a cycle on n vertices by adding a new vertex and making this vertex adjacent to all vertices on the cycle.

Notice that the smallest wheel W_3 is K_4. We denote the class of wheels by $\mathcal{W}$.

Definition 1.2. Let G be a graph. If u, v are nonadjacent vertices of G, then $G + uv$ is obtained from G by *adding* a new edge uv.

Definition 1.3. If v has a degree at least four, then by *splitting* v we mean the operation of first deleting v from G, then *adding* two new adjacent vertices v', v'' and joining each neighbor of v to exactly one of v', v'' such that each of v', v'' has degree at least three in the new graph.

Definition 1.4. If v has degree at least four, then 3-*splitting* v means *splitting* v such that either v' or v'' has degree 3.

The next is a classical result of Tutte, which explains how 3-connected graphs are generated.

Theorem 1.5. *(Tutte's Wheel Theorem)* [14] *A non-wheel graph G is 3-connected if and only if G is constructed from a wheel by repeatedly adding edges and splitting vertices.*

The theorem was improved such that the starting graph needed was only W_4.

Theorem 1.6. *(Ding and Qin)* [4] *A non-wheel graph G is 3-connected if and only G is constructed from W_4 by repeatedly adding edges and splitting vertices.*

We further improved the theorem by restricting the vertex splitting operation to 3-splitting, which reduced the computation time for generating 3-connected graphs from W_4.

Theorem 1.7. *(S., Ding, 2023^+) A non-wheel graph G is 3-connected if and only if G is constructed from W_4 by repeatedly adding edges and 3-splitting vertices.*

2. Generalized Chain Theorem On Rooted graphs. For a 3-connected graph, we were interested in scenarios where the two operations of *adding* an edge and *3-splitting* a vertex are avoided on a set $R \subseteq V(G)$.

Definition 2.1. A *separator* S of a graph G is a subset of $V(G)$ such that $G\backslash S$ is disconnected.

Definition 2.2. A k-*separator* for a connected graph G is a subset S of $V(G)$ such that $|S| = k$ and $G\backslash S$ is disconnected.

Our motivation was to examine the components of $G\backslash S$ for a 3-connected graph G over a 3-*separator* S. This would allow us to analyze whether a particular minor is possible. A *rooted graph* is a pair (G, R), where G is a graph and R is a set of vertices of G with $0 < |R|$ and $G[R]$ is a clique. A Δ-*graph* is a rooted graph (G, R) such that G is 3-connected, and $G\backslash R$ is connected.

For any integers $n > 2$ and $k > 3$, let $W_n \bigoplus_3 K_k$ denote the graph obtained by identifying a traingle of W_n with a traingle of K_k. For any integers $n > 2$ and $k \geq 0$, let $\mathcal{W}_n^k$ be the rooted graph (G, R), where if $k > 3$, then $G = W_n \bigoplus_3 K_k$ and $R = V(K_k)$. If $k \leq 3$, then $G = W_n$ and R is a set of vertices of k-clique of G.

Theorem 2.3. *(S., Ding,2024^+) A rooted graph (G, R), where $|R| = k$, is a Δ-graph if and only if $G = \mathcal{W}_n^k$ or (G, R) is constructed from $(\mathcal{W}_4^k, V(K_k))$ by repeatedly adding edges and 3-splitting vertices.*

Note that Theorem 1.7 is the special case of Theorem 2.3 whenever $k = 1$.

3. Chain Theorem on Smoothly 3-Connected Graphs. Next, we were interested in 3-connected graphs with a special property.

Definition 3.1. Let G be a 3-connected graph with $\mathcal{S}_3$ the set of *separators* of size 3. G is *smoothly* 3-connected(*S3C*) if for every $S \in \mathcal{S}_3$, $G\backslash S$ has exactly two components.

We were interested in having a chain theorem for the class of *S3C* graphs. Having such a chain theorem would guarantee that all graphs in the class could be constructed from W_4 while keeping each of the intermediate constructions also in the class of $S3C$ graphs.

Definition 3.2. A 3-split on a S3C graph G is called *non-smooth* if the resulting graph is no longer S3C. A 3-split that preserves S3C is called *smooth*.

Theorem 3.3. *A graph G is smoothly 3-connected if and only if G is a wheel or G is constructed from W_4 by repeatedly performing the two operations of adding an edge and applying a smooth 3-split on a vertex.*

C. Generalization of a Block Tree Decomposition

We aimed to examine *S-fragments* of 3-connected graphs over 3-*separators* and sought a tree structure that captures the decomposition of the graph into its connected components and *separators*. There has been plenty of work on graph decomposition, for example, Clique tree decomposition, atom tree decomposition, and chordal tree decomposition. (See [1, 8, 12, 13]). We wanted a generalization that would also apply to all the previously formulated decompositions. We first define a few terms.

Definition 3.4. A set $X \subseteq V(G)$ is *separated* by a *separator* S if at least two components of $G\backslash S$ contain vertices of X.

Remark. By the above definition, any subset of S is not *separated* by S.

Definition 3.5. Let $G_1, \ldots, G_k$ $(k \geq 2)$ be the connected components of $G\backslash S$. Then we define *S-fragments* of G as $F_i = G[S \cup V(G_i)]$ $(i = 1, 2, \ldots, k)$ with all possible edges added between vertices in S.

Definition 3.6. Let $G_1, \ldots, G_k (k \geq 2)$ be the components of $G\backslash S$. Then S is called *genuine* if $N_G(V(G_i)) = S$ holds for at least two components. Then, if $N_G(V(G_i)) = S$, we call G_i a *genuine* component of S and F_i a *genuine S-fragment* of G.

Definition 3.7. Let $\mathcal{S}$ be a set of *separators* of G such that no *separator* in $\mathcal{S}$ is separated by any other *separator* in $\mathcal{S}$. Then, we call $\mathcal{S}$ a set of *non-crossing separators.*

Definition 3.8. A *separator* S of a graph G is called a *clique separator* if $G[S]$ is a complete graph.

4. Layout Tree.

Definition 4.1. Let $G = (V, E)$ be connected and $\mathcal{S}$ be a set of *separators* of G. We say that a labeled tree T is a *layout tree* of $(G, \mathcal{S})$ if it satisfies the following two properties:

- (T1) $V(T) \supseteq \mathcal{S}$ is a set of subsets of V such that $\cup\{X : X \in V(T)\} = V(G)$.
- (T2) for each $S \in \mathcal{S}$, if $T_1, \ldots, T_k$ are all the $\{S\}$-fragments of T, then $k \geq 2$, and for each $i \in \{1, \ldots, k\}, G_i := \cup\{G[X] : X \in V(T_i)\}$ is a union of S-fragments of G; moreover, $V(G_i \cap G_j) = S$ holds for all distinct i, j.

We call a *layout tree* of $(G, \mathcal{S})$ *strong* if it holds the property:

- (T2*) the collection G_i in $(T2)$ are all the S-fragments of G.

Lemma 4.2. *If there exists a layout tree of $(G, \mathcal{S})$, then there exists a layout tree T that also satisfies the following properties:*

- *(T3) $V(T) - \mathcal{S}$ is stable in T.*
- *(T4) $\mathcal{S}$ is stable in T.*
- *(T5) If $SX \in E(T)$ with $S \in \mathcal{S}$ and $X \notin \mathcal{S}$, then $S \subseteq X$.*
- *(T6) If $S \in \mathcal{S}$, then $d_T(S) = 2$.*

Moreover, if there exists a layout tree of $(G, \mathcal{S})$ that is strong, then there exists a strong layout tree T satisfying $(T3 - 6)$.

Theorem 4.3. *Let $G = (V, E)$ be a connected graph, and let $\mathcal{S}$ be a set of genuine non-crossing separators of G. Then, a layout tree T exists for $(G, \mathcal{S})$. Moreover, if the members of $\mathcal{S}$ are all minimal, then a strong layout tree T exists for $(G, \mathcal{S})$.*

Proposition 4.4. *Let S_1, S_2 be 3-separator of 3-connected graph G. If S_1 is not smooth, then S_1 does not separate S_2.*

Lemma 4.5. *The collection of all non-smooth 3-separators of a graph G is a set of genuine non-crossing separators.*

Theorem 4.6. *Let G be a 3-connected graph and let $\mathcal{S}$ be a collection of non-smooth 3-separators in G. There exists a strong layout-tree of $(G, \mathcal{S})$. Moreover, if $\mathcal{S}$ contains all the non-smooth 3-separators in G, then, each non-separator node in the strong layout-tree corresponds to a unique smoothly 3-connected minor of G.*

D. Applications

For a graph H, we denote $\mathcal{F}(H)$ as the class of all H-free 3-connected graphs. Prism-free graphs have been characterized. Let $\mathcal{K}$ be the class of 3-connected graphs G for which there exists a set X of three vertices such that $G\backslash X$ is edgeless. Equivalently, such a graph G is obtained from $K_{3,n}$ ($n \geq 1$) by adding edges to its color class of size three. The following result, which characterizes prism-free graphs, can also be proved using Theorem 2.3 and Theorem 3.3 established above.

Theorem 4.7. *(Dirac 1963, Lovasz 1965).* [5, 9] $\mathcal{F}(Prism) = \{K_5\} \cup \mathcal{W} \cup \mathcal{K}$.

Let $K_5^{\perp}$ denote the graph obtained by 3-*splitting* a vertex in K_5. The following result can be proved using the above two theorem along with the Sagemath program to find the maximal *smoothly* 3-connected graphs.

Theorem 4.8. *(Oxley 1989) $\mathcal{F}(W_5)$ consists of $\mathcal{K}$ and 3-connected minors of graphs in {Cube, Octahedron, Pyramid, $K_5^{\perp}$}.*

E. Future Works

5. Characterization of H-free graphs for large minors of Petersen Graph. In graph theory, H-free graphs are central to several longstanding open problems. For instance, Hadwiger's Conjecture (1943) states that every K_n-free graph is $n-1$ colorable. Today, this conjecture remains "one of the deepest unsolved problems in graph theory".

Another longstanding problem of this kind is Tutte's 4-flow conjecture, which asserts that every bridgeless *Petersen*-free graph admits a 4-flow. It is generally believed that knowing the structures of K_n-free graphs and *Petersen*-free graphs, respectively, would lead to a solution to the corresponding conjecture.

In their Graph-Minors project, Robertson and Seymour obtained, for every graph H, an approximate structure for H-free graphs. However, this result is insufficient to address the conjectures mentioned above. Note that both K_6 and *Petersen* graph have fifteen edges. Currently, there is no connected graph H with that many edges for which H-free graphs are completely characterized.

There are known results for minors of *Petersen* graph that are of sizes 12 and 13. (See [6]). The next step is to characterize H-free graphs for a *Petersen* minor of size 14, with the goal of extending this approach to K_6. We aim to gradually characterize H-free graphs for increasingly larger minors, ultimately including graphs with 15 edges such as K_6 and the *Petersen* graph.

6. More Chain Theorem. The current motivation is to find more chain theorems to understand the structures of various classes of 3-connected graphs. One of the particular interest is on simplifying the known result of $(Oct\backslash e)$-free graphs [3].

A 3-*sum* of two 3-connected graphs G_1, G_2 is obtained by identifying a triangle of G_1 with a triangle of G_2. Some common edges could be deleted after the identification, as long as no degree-two vertices are created. It is not difficult to verify that the resulting graph is always 3-connected.

Let $\mathcal{S}$ be the set of graphs obtained by 3-summing wheels and Prisms over a common triangle. In other words, every graph in $\mathcal{S}$ is constructed from a set of wheels and Prisms, each with a specified triangle, by identifying all these specified triangles. Edges of these triangles could be deleted after the identification. It is worth pointing out that every 3-connected minor of a graph in $\mathcal{S}$ remains in $\mathcal{S}$. Because 3-connected minors of a wheel are till wheels and 3-connected minors of a Prism are also wheels. The class $\mathcal{S}$ is contained in $\mathcal{F}(Oct\backslash e)$.

Another class that will help simplify the result is a class of *smoothly* 3-connected graphs with an added property that if S is a 3-*separator* for a *smoothly* 3-connected graph G, then neither of the two S-*fragments* of G is a W_n $(n \geq 5)$. We will denote this class as $\mathcal{T}$. The class $\mathcal{T}$ has graphs that are close to graphs in $\mathcal{S}$. For example, if $G \in \mathcal{S}$ and G is a 3-*sum* of a large wheel and a prism, then the graph H obtained by *adding* an edge to the wheel is in the class $\mathcal{T}$. Graphs similar to H contain $Oct\backslash e$ as a minor. Hence, having a chain theorem for the class $\mathcal{T}$ would help make a conclusion regarding $\mathcal{S}$. It can then be concluded that any other 3-connected graphs constructed from a graph in $\mathcal{S}$ by *adding* and edge to a wheel must contain $Oct\backslash e$ as a minor. The rest of the finite graphs in $\mathcal{F}(Oct\backslash e)$ can be determined by computer construction.

F. Conclusion

In conclusion, using the chain theorems and computer program for minor testing, I have demonstrated simpler ways of characterizing $\mathcal{F}(H)$ for graphs H of edge sizes up to 12.

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