The conclusion from the foregoing is that Steinhaus conjectured the ham sandwich theorem and Banach gave the first proof, using the Ulam-Borsuk theorem. This shows that Stone and Tukey were not correct in attributing the ham sandwich theorem to Ulam. However, Ulam did make a fundamental contribution in proposing the antipodal map theorem.

Remarks. We first mention a recent application by Blair Swartz of ham sandwich theorems for fractions other than $1/2$ to interface reconstruction in hydrodynamic calculations. See paragraph 20 of the web site:

http://www-troja.fjfi.cvut.cz/~liska/bbw/abs-list.html

There is a cautionary note stating that for some shapes or configurations of cells there exist n-tuples of mass fractions that cannot be simultaneously sliced from cells.

Finally, we note a paper by Steinhaus [**3**] that represents work Steinhaus did in Poland on the ham sandwich problem in World War II while hiding out with a Polish farm family.

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Los Alamos National Laboratory, Mail Stop K710, Los Alamos, NM, 87545
beyer@lanl.gov
azz@lanl.gov

Roots Appear in Quanta

Alexander R. Perlis

We start with a special case. Consider an irreducible quintic polynomial

$$f(X) = X^5 + a_1X^4 + a_2X^3 + a_3X^2 + a_4X + a_5$$

with rational coefficients and with three real roots and one pair of complex conjugate roots. For example, $f(X)$ could be $X^5 - 10X + 5$.

Question. If α is a root of f, then how many roots of f lie in the field $\mathbf{Q}(\alpha)$?

The field $\mathbf{Q}(\alpha)$ is obtained by adjoining the root α to $\mathbf{Q}$. Thus $\mathbf{Q}(\alpha)$ contains at least one root of f, and of course it can contain at most five roots of f.

Answer. The number $r(f)$ of roots of f in $\mathbf{Q}(\alpha)$ is 1. We prove that, for an arbitrary irreducible polynomial f and root α, $r(f)$ divides the degree of f. For the quintic under discussion, adjoining one of the real roots cannot possibly produce the nonreal roots, so $r(f)$, being a divisor of 5, must be 1.

An informal survey of books and colleagues indicates that the divisibility result "$r(f)$ divides the degree" is not well known. In what follows, K is a field and, unless stated otherwise, all roots and field extensions are taken in a fixed algebraic closure $\overline{K}$ of K. When $K = \mathbf{Q}$, we always take $\overline{K}$ inside the complex numbers so that we can speak of real roots and nonreal roots.

Theorem 1. *Let $f(X)$ in $K[X]$ be an irreducible polynomial, and let α be a root of f. Set*

$$r_K(f) := \textit{number of roots of } f \textit{ that lie in } K(\alpha),$$

$$s_K(f) := \textit{number of fields of the form } K(\alpha')\textit{, where } \alpha' \textit{ is a root of } f.$$

Then $r_K(f)$ is independent of the choice of α, and

$$r_K(f) \cdot s_K(f) = \textit{cardinality of the set of roots of } f.$$

In particular, $r_K(f)$ divides the degree of f.

Concerning the last statement of the theorem: the cardinality of the set of roots of f is known as the *separable degree* of f, and it is well known that the separable degree divides the usual degree.

Proof. For this proof, we let "root" mean "root of f" and let "stem field" signify a field of the form $K(\alpha')$, where α' is a root. Since f is irreducible, each stem field is K-isomorphic to the abstract field $K[X]/\big(f(X)\big)$, whence any two stem fields are K-isomorphic. Isomorphisms take roots to roots, so $r_K(f)$ is the same for each stem field. Each root α' lies in precisely one stem field: it lies in $K(\alpha')$, and if it also lies in $K(\alpha'')$, then $K(\alpha') \subseteq K(\alpha'')$, but because the two stem fields have the same degree over K (they are K-isomorphic), we must have $K(\alpha') = K(\alpha'')$. In summary, the set of roots is partitioned by the stem fields into $s_K(f)$ collections with $r_K(f)$ roots in each collection, making $r_K(f) \cdot s_K(f)$ the cardinality of the set of roots. ■

The symbol $r_K(f)$ is determined both by the polynomial f and by the base field K. When K is understood, as it was earlier when $K = \mathbf{Q}$, the simpler notation $r(f)$ can be used. There doesn't seem to be an established name for the quantity $r_K(f)$, and I propose: *root quantum number of f over K*. While this name initially sounds rather fancy for a simple concept, the following theorem shows that the roots of f really do come bundled in collections of size $r_K(f)$.

Theorem 2. *Let $f(X)$ in $K[X]$ be irreducible. If L/K is a field extension (not necessarily algebraic), then the number of roots of f in L is a multiple of $r_K(f)$.*

Proof. The proof of Theorem 1 exhibits a partition of the set of roots of f into collections of equal size $r_K(f)$, where each collection has the property: in any field extension of K, the presence of one of the roots implies the presence of the remaining ones. ■

Remark. We can also see that the cardinality of the set of roots of f lying *outside* a given extension L/K (counted in an algebraically closed field containing L) is a multiple of $r_K(f)$. Theorem 1 shows that $r_K(f)$ divides the total number of roots, and Theorem 2 shows that $r_K(f)$ divides the number of roots in L, so $r_K(f)$ also divides the difference of these two numbers.

Corollary. *If $f(X)$ in $\mathbf{Q}[X]$ is irreducible, then the number of real roots of f is a multiple of $r_{\mathbf{Q}}(f)$. The same can be said about the number of nonreal roots.*

Proof. Keeping the remark in mind, take $L = \mathbf{R}$ in Theorem 2. ∎

Theorem 2 may be summarized as follows: *roots appear in quanta*. This places combinatorial restrictions on the way f can factor. For example, if $f(X)$ in $K[X]$ is irreducible and separable of degree 15, with α a root, then the factorization of f over $K(\alpha)$ cannot have the following form:

$$\text{(linear)(linear)(linear)(quadratic)(quadratic)(octic)}.$$

To see this, assume for the sake of contradiction that the factorization of f over $K(\alpha)$ has the form indicated. Since f is separable, the three linear factors correspond to distinct roots of f in $K(\alpha)$, so $r_K(f) = 3$. The field L obtained from $K(\alpha)$ by adjoining the roots of the two quadratic factors has degree at most 4 over $K(\alpha)$. Thus L contains none of the roots of the octic factor, so L contains precisely seven of the roots of f. This contradicts the fact that the number of roots of f in L must be a multiple of three.

The interested reader can check that the root quantum number has the following three descriptions in terms of Galois theory. Let f be irreducible and separable over K, with Galois group G, viewed as a permutation group on the set of roots of f. Let $H \subset G$ be the subgroup fixing a root α. Then:

i. $r_K(f)$ is the number of roots fixed by H;
ii. $r_K(f)$ is the cardinality of $\mathrm{Aut}(K(\alpha)/K)$; and
iii. $r_K(f)$ is the index $[\mathrm{N}_G(H) : H]$ of H in its normalizer.

Finally, it is instructive to think about the triples (K, n, r) that indicate the existence of an irreducible polynomial $f(X)$ in $K[X]$ of degree n with root quantum number r. The necessary condition discussed in this note is that r must divide n. Here are some exercises involving these triples:

1. Show that $(\mathbf{Q}, 2, 1)$ does *not* appear.
2. Find a field K for which $(K, 2, 1)$ *does* appear.
3. Let r divide n. Show that there exists K for which (K, n, r) appears.
4. (Advanced) Let r divide n. Except for $(\mathbf{Q}, 2, 1)$, show that $(\mathbf{Q}, n, r)$ appears.

Solutions can be obtained from the author.

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Department of Mathematics, The University of Arizona, Tucson, Arizona, 85721–0089
aprl@math.arizona.edu