Heronian tetrahedra are lattice tetrahedra

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Abstract

Extending a similar result about triangles, we show that each Heronian tetrahedron may be positioned with integer coordinates. More generally, we show the following: if an integral distance set in $\mathbf{R}^3$ can be positioned with rational coordinates, then it can in fact be positioned with integer coordinates. The proof, which uses the arithmetic of quaternions, is tantamount to an algorithm.

1 Motivations.

From P. Yiu in this MONTHLY [8] we learned that Heronian triangles can be realized as lattice triangles; that is, *each triangle with integer side lengths and integer area can be positioned in the plane so that its three vertices have integer coordinates.* Yiu includes this example: the triangle with side lengths $(25, 34, 39)$, which has area 420, can be realized with integer coordinates as shown in Figure 1, despite the fact that this triangle has no integer heights.

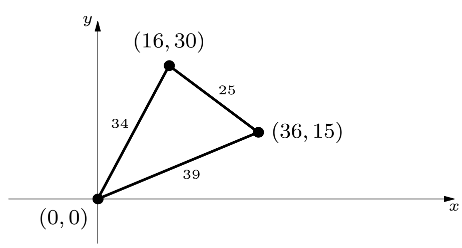


Figure 1: Heronian triangle $(25, 34, 39)$ as a lattice triangle

We wondered whether a similar result holds for Heronian tetrahedra and asked the following. *Can each tetrahedron with integer side lengths, integer face areas, and integer volume be positioned in three-dimensional space so that its four vertices have integer coordinates?* The classification of Heronian tetrahedra is incomplete (see [2] and its references for a recent state of affairs); however, using the formulas in [1] for computing the face areas and volume of a tetrahedron, we used a computer to determine all Heronian tetrahedra with side length up to 34000, and in each case the computer was able to find a position with integer coordinates. For example, the computer found integer coordinates for the Heronian tetrahedron in Figure 2, which has face areas 6300, 4914, 2436, 3570, and volume 35280.

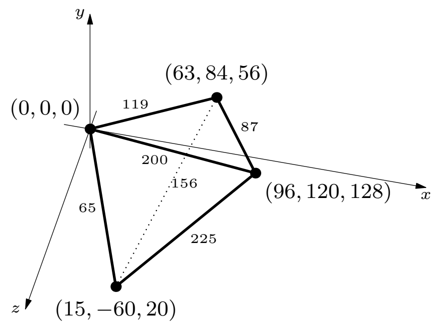


Figure 2: Tetrahedron $(225, 200, 65, 119, 156, 87)$ as a lattice tetrahedron

To answer our question for *all* Heronian tetrahedra, our initial hope was to adapt Yiu's method. Yiu considers the Heron triangle area formula from the viewpoint of solving for one of the edges, and deftly manipulates a corresponding discriminant condition into a form whose integral solutions are known to be the precise sums of squares needed to obtain integer coordinates. However, we subsequently learned from J. Fricke [4] another method for showing Heronian triangles to be lattice triangles. Using the arithmetic of Gaussian integers combined with their interpretation as two-dimensional rotations, Fricke provides a constructive algorithm for rotating such a triangle into a position with integer coordinates. Guided by that method, we were able to show Heronian tetrahedra to be lattice tetrahedra. Using the arithmetic of Lipschitz-integral quaternions combined with their interpretation as three-dimensional rotations, we obtained a constructive algorithm for rotating such a tetrahedron into a position with integer coordinates.

We shall review Fricke's method, then cover the quaternion method, and finally show how to recover the former as a special case of the latter.

2 Fricke's method.

Fricke views Heronian triangles as particular examples of *integral distance sets*; that is, collections of points whose pairwise distances are integers. He then shows Heronian triangles to be lattice triangles for an elegantly simple reason:

1. Heronian triangles are easily positioned with rational coordinates; and
2. *each finite integral distance set in* $\mathbf{Q}^2$ *can be repositioned to lie in* $\mathbf{Z}^2$.[‡]

[‡]For our purposes here, we are not concerned with *infinite* integral distance sets, but note, for the sake of completeness, that the infinite case follows from the Anning–Erdős Theorem (elegant proof in [3]): *if* $M \subset \mathbf{Q}^n$ *is an infinite set of points such that each mutual distance is an integer, then* M *lies in a line, and thus can be repositioned to lie in* $\mathbf{Z}^1$.

By the area formula for triangles, $\Delta = \frac{1}{2} \cdot \text{base} \cdot \text{height}$, Heronian triangles have rational heights, so the first statement is verified by Figure 3. The key to Fricke's

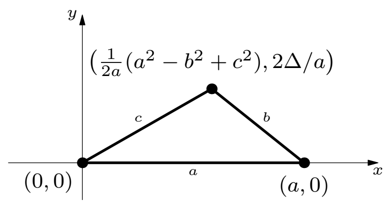


Figure 3: Rational coordinates for Heronian triangle (a, b, c) with area Δ

method is the second statement. Since we can translate an integral distance set by one of its points, there is no loss of generality in assuming the integral distance set to include the origin, and the second statement becomes as follows.

Theorem 1. *Let $M \subset \mathbf{Q}^2$ be a finite set of rational points with $(0,0) \in M$ such that each mutual distance is an integer. Then there exists a rotation T such that $TM \subset \mathbf{Z}^2$.*

Instead of rotating directly from rational to integer coordinates, we avoid denominators as follows. We first scale up to clear all denominators, then rotate into integer coordinates *that are also multiples of our scaling factor*, and then scale back down. By writing the scaling factor as a product of primes, we can even work on one prime at a time. We claim Theorem 1 to be equivalent to the following.

Theorem 2. *Let p be a prime. Let $M \subset \mathbf{Z}^2$ be a finite set of points with $(0,0) \in M$ such that each mutual distance is an integer divisible by p. Then there exists a rotation T such that $TM \subset p\mathbf{Z}^2$.*

Proof of equivalence. Assume Theorem 1 holds. To establish Theorem 2, start with the set M from Theorem 2. Now apply Theorem 1 to the set $\frac{1}{p}M$ to obtain $T(\frac{1}{p}M) \subset \mathbf{Z}^2$, whence $TM \subset p\mathbf{Z}^2$, thus giving us Theorem 2.

For the other direction, assume Theorem 2 holds. Let $M \subset \mathbf{Q}^2$ be a finite set of points such that each mutual distance is an integer. Let d be such that $dM \subset \mathbf{Z}^2$. Let $d = p_1 \cdots p_n$ be a factorization into primes (repetition allowed). All distances in dM are divisible by p_1, so by Theorem 2 there exists T_1 with $T_1(dM) \subset p_1\mathbf{Z}^2$, or $T_1(\frac{d}{p_1}M) \subset \mathbf{Z}^2$. All distances in $\frac{d}{p_1}M$ are divisible by p_2, so by Theorem 2 there exists T_2 with $T_2(T_1(\frac{d}{p_1}M) \subset p_2\mathbf{Z}^2$, or $T_2(T_1(\frac{d}{p_1 p_2}M) \subset \mathbf{Z}^2$. Continuing in this fashion, we obtain

$$T_n(\cdots T_2(T_1(\tfrac{d}{p_1 p_2 \cdots p_n}M)) \cdots) \subset \mathbf{Z}^2,$$

or simply $(T_n \cdots T_2 T_1)M \subset \mathbf{Z}^2$, thus giving us Theorem 1. $\square$

To prove Theorem 2, we will represent points and rotations using Gaussian integers. Recall that the ring $\mathbf{Z}[\mathbf{i}]$ of Gaussian integers comprises complex numbers of the form $a + b\mathbf{i}$, where a and b are (ordinary) integers. The *conjugate* of

$\mathbf{u} = a + b\mathbf{i}$ is $\overline{\mathbf{u}} = a - b\mathbf{i}$ and the *norm* of $\mathbf{u} = a + b\mathbf{i}$ is $\mathrm{N}(\mathbf{u}) = \overline{\mathbf{u}}\mathbf{u} = a^2 + b^2$. Note that $\mathrm{N}(\mathbf{u})$ is the square of the length of $\mathbf{u}$; hence, under the identification $(x, y) \leftrightarrow x + y\mathbf{i}$ between points in the plane with integer coordinates and Gaussian integers, statements involving distances can be translated into statements involving norms. A nonzero Gaussian integer π gives rise to the operator

$$T_\pi \colon (x, y) \mapsto \tfrac{\overline{\pi}}{\pi}(x + y\mathbf{i}), \tag{1}$$

which is a rotation because the complex number $\overline{\pi}/\pi$ has norm 1. To find the appropriate rotation operator to prove Theorem 2, we first need some preliminaries on the arithmetic of Gaussian integers.

Just as each number in $\mathbf{Z}$ can be factored into an essentially unique product of primes (unique up to order and multiplication by units), the same is true for numbers in $\mathbf{Z}[\mathbf{i}]$, where the set of units is $\{\pm 1, \pm\mathbf{i}\}$. However, $\mathbf{Z}$ and $\mathbf{Z}[\mathbf{i}]$ have different sets of primes: $p = 2$ factors inside $\mathbf{Z}[\mathbf{i}]$ as $\mathbf{i}(1 - \mathbf{i})^2$, $p \equiv 3 \pmod 4$ remains prime inside $\mathbf{Z}[\mathbf{i}]$, and $p \equiv 1 \pmod 4$ splits inside $\mathbf{Z}[\mathbf{i}]$ as $\overline{\pi}\pi$, where π and $\overline{\pi}$ are the two primes (up to units) of norm p in $\mathbf{Z}[\mathbf{i}]$. To learn more about Gaussian integers, see [7, Chapter 14].

Lemma 3. *Given $\mathbf{u} \in \mathbf{Z}[\mathbf{i}]$, if p is an odd prime such that $p \nmid \mathbf{u}$ but $p \mid \mathrm{N}(\mathbf{u})$, then $\mathbf{u}$ has a unique divisor of norm p (unique up to units).*

Proof. From $p \nmid \mathbf{u}$ follows $p \nmid \overline{\mathbf{u}}$, yet $p \mid \overline{\mathbf{u}}\mathbf{u}$, whence p is not prime in $\mathbf{Z}[\mathbf{i}]$, so $p \equiv 1 \pmod 4$. Thus $p = \overline{\pi}\pi$. Now $\pi \mid \overline{\mathbf{u}}\mathbf{u}$ while $\overline{\pi}\pi \nmid \mathbf{u}$, so π divides precisely one of $\mathbf{u}, \overline{\mathbf{u}}$ (and $\overline{\pi}$ divides the other). $\square$

Lemma 4. *Given $\mathbf{u}, \mathbf{u}' \in \mathbf{Z}[\mathbf{i}]$, if p is an odd prime such that p divides neither $\mathbf{u}$ nor $\mathbf{u}'$, but p divides both $\mathrm{N}(\mathbf{u})$ and $\mathrm{N}(\mathbf{u}')$, then $\mathbf{u}$ and $\mathbf{u}'$ share a divisor of norm p if and only if $p \mid \mathrm{N}(\mathbf{u} - \mathbf{u}')$.*

Proof. As before, we must have $p \equiv 1 \pmod 4$. Write $p = \overline{\pi}\pi$, labeled so that $\pi \mid \mathbf{u}$. Note $\pi \nmid \overline{\mathbf{u}}$. If $\pi \mid \mathbf{u}'$ then $\pi \mid \mathbf{u} - \mathbf{u}'$ whence $p \mid \mathrm{N}(\mathbf{u} - \mathbf{u}')$, while if $\pi \nmid \mathbf{u}'$ then $\pi \nmid \mathrm{N}(\mathbf{u} - \mathbf{u}') = \mathrm{N}(\mathbf{u}) + \mathrm{N}(\mathbf{u}') - \mathbf{u}\overline{\mathbf{u}'} - \overline{\mathbf{u}}\mathbf{u}'$ whence $p \nmid \mathrm{N}(\mathbf{u} - \mathbf{u}')$. $\square$

Remark. Since all Gaussian integers of norm 2 differ by a unit, one can show that Lemma 3 and Lemma 4 hold also for $p = 2$, but we do not need that case.

Lemma 5. *Let p be a prime. If $\mathbf{u} \in \mathbf{Z}[\mathbf{i}]$ satisfies $p \nmid \mathbf{u}$ but $p^2 \mid \mathrm{N}(\mathbf{u})$, then $\mathbf{u}$ has a factorization of the form $\mathbf{u} = \pi^2\mathbf{s}$ for a unique π of norm p (unique up to units). Furthermore, if $\mathbf{u}' \in \mathbf{Z}[\mathbf{i}]$ satisfies the same hypotheses as $\mathbf{u}$, then $\mathbf{u}'$ has a factorization $\mathbf{u}' = \pi^2\mathbf{s}'$ (where π is the same as for $\mathbf{u}$) if and only if $p \mid \mathrm{N}(\mathbf{u} - \mathbf{u}')$.*

Proof. We must have p odd since $2 \nmid \mathbf{u}$ is incompatible with $\mathrm{N}(\mathbf{u}) \equiv 0 \pmod 4$. By Lemma 3, $\mathbf{u}$ has a unique divisor π of norm p. Write $\mathbf{u} = \pi\mathbf{w}$. Now $p \nmid \mathbf{w}$ but $p \mid \mathrm{N}(\mathbf{w})$. By Lemma 3, $\mathbf{w}$ has a unique divisor of norm p, which is then a divisor of $\mathbf{u}$, thus must be π. For the second part, apply Lemma 4. $\square$

We are now ready to prove Theorem 2.

Proof of Theorem 2. By replacing distance statements with norm statements under the identification $(x, y) \leftrightarrow x + y\mathbf{i}$, we get that M is a set of Gaussian integers satisfying

$$p^2 \mid \mathrm{N}(\mathbf{u}) \quad \text{and} \quad p^2 \mid \mathrm{N}(\mathbf{u} - \mathbf{u}'), \quad \text{for all } \mathbf{u}, \mathbf{u}' \in M.$$

If $p \mid \mathbf{u}$ for all $\mathbf{u} \in M$, then already $M \subset p\mathbf{Z}^2$. Otherwise, there exists $\mathbf{u}$ with $p \nmid \mathbf{u}$. By Lemma 5, there exists π of norm p with $\pi^2 \mid \mathbf{u}$. Let $\mathbf{u}' \in M$. If $p \nmid \mathbf{u}'$, then Lemma 5 tells us that $\pi^2 \mid \mathbf{u}'$. If $p \mid \mathbf{u}'$, then $p \nmid \mathbf{u} - \mathbf{u}'$. Hence, Lemma 5 tells us that $\pi^2 \mid \mathbf{u} - \mathbf{u}'$, whence $\pi^2 \mid \mathbf{u}'$. Either way, π^2 divides every point in M; therefore, $T_\pi M \subset p\mathbf{Z}^2$, where T_π is the rotation operator (1). $\square$

3 Example of Fricke's method.

We wish to realize the Heronian triangle $(65, 17, 80)$, which has area 288, as a lattice triangle. Under the identification $(x, y) \leftrightarrow x + y\mathbf{i}$, we position the triangle initially with rational coordinates, and then scale up by $5 \cdot 13$ to clear denominators, as shown in Figure 4. The scaled vertices satisfy the hypotheses of Theorem 2 for $p = 5$ and $p = 13$.

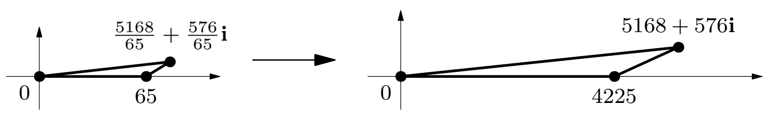


Figure 4: Heronian triangle $(65, 17, 80)$, positioned initially with rational coordinates, then scaled up to lattice triangle $5 \cdot 13 \cdot (65, 17, 80)$

We first consider $p = 5$. We observe $5 \nmid 5168 + 576\mathbf{i}$ but $5 \mid \mathrm{N}(5168 + 576\mathbf{i})$. Thus we seek the unique Gaussian prime π of norm 5 that divides $5168 + 576\mathbf{i}$, whose existence is guaranteed by Lemma 3. We note that checking $\pi \mid 5168 + 576\mathbf{i}$ is equivalent to checking $5 \mid (5168 + 576\mathbf{i})\overline{\pi}$. To find the candidate primes of norm 5, we write $5 = 2^2 + 1^2 = (2 + \mathbf{i})(2 - \mathbf{i})$, and conclude $\pi = 2 - \mathbf{i}$ by verifying

$$(5168 + 576\mathbf{i}) \cdot \overline{2 - \mathbf{i}} = 5 \cdot (1952 + 1264\mathbf{i}).$$

Lemma 5 ensures $(2 - \mathbf{i})^2$ divides all coordinates, allowing us to rotate by $T_{2-\mathbf{i}}$; that is, we multiply all coordinates by $(2+\mathbf{i})/(2-\mathbf{i})$. The result is a new position in which all coordinates are divisible by 5, allowing us to scale down by 5, as shown in Figure 5.

Next consider $p = 13$. The vertices of our scaled and rotated triangle still satisfy Theorem 2 for $p = 13$ and we observe $13 \nmid 528 + 896\mathbf{i}$; thus, we seek the unique Gaussian prime of norm $13 = (2 + 3\mathbf{i})(2 - 3\mathbf{i})$ that divides $528 + 896\mathbf{i}$, and it turns out to be $2 + 3\mathbf{i}$:

$$(528 + 896\mathbf{i}) \cdot \overline{2 + 3\mathbf{i}} = 13 \cdot (288 + 16\mathbf{i}).$$

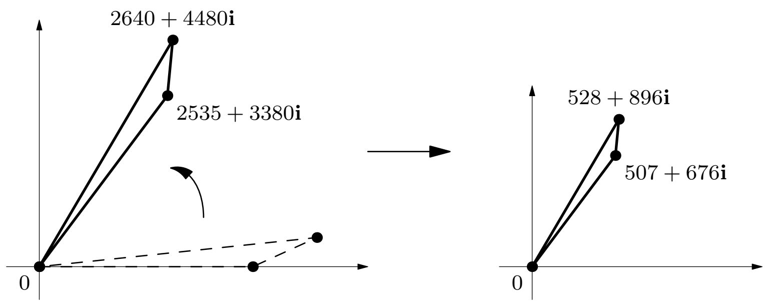


Figure 5: Lattice triangle $5 \cdot 13 \cdot (65, 17, 80)$ first rotated by $T_{2-\mathbf{i}}$ and then scaled down to lattice triangle $13 \cdot (65, 17, 80)$

We now rotate by $T_{2+3\mathbf{i}}$; that is, we multiply all coordinates by $(2-3\mathbf{i})/(2+3\mathbf{i})$. As all coordinates were divisible by $(2+3\mathbf{i})^2$, the result is a new position in which all coordinates are divisible by 13, allowing us to scale down by 13, as shown in Figure 6. The final result is the original triangle $(65, 17, 80)$ realized as a lattice triangle.

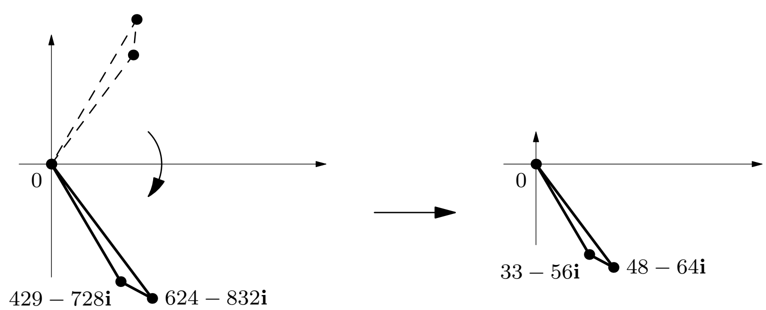


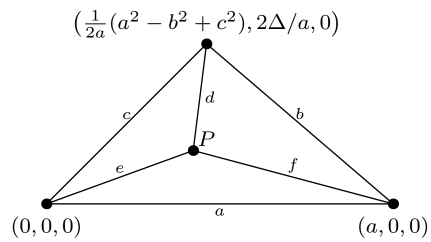
Figure 6: Lattice triangle $13 \cdot (65, 17, 80)$ first rotated by $T_{2+3\mathbf{i}}$ and then scaled down to the original triangle $(65, 17, 80)$ now positioned as a lattice triangle

4 Tetrahedra.

Analogously to Fricke's method, we show that each Heronian tetrahedron is a lattice tetrahedron by establishing the following:

1. Heronian tetrahedra are easily positioned with rational coordinates; and
2. *each finite integral distance set in* $\mathbf{Q}^3$ *can be repositioned to lie in* $\mathbf{Z}^3$.

By the volume formula for tetrahedra, $V = \frac{1}{3} \cdot \text{base face area} \cdot \text{height}$, Heronian tetrahedra have rational heights, so the first statement is verified by Figure 7. As in the two-dimensional case, the key is the second statement, which we state more precisely as follows.



$$P = \Big(\tfrac{1}{2a}(a^2-f^2+e^2),\, \tfrac{1}{8a\Delta}\big((b^2-c^2)(e^2-f^2) + a^2(b^2+c^2-2d^2+e^2+f^2) - a^4\big), 3V/\Delta\Big),$$

where Δ is the area of face abc and V is the volume of the tetrahedron.

Figure 7: Heronian tetrahedron positioned with rational coordinates

Theorem 6. *Let $M \subset \mathbf{Q}^3$ be a finite set of rational points with $(0,0,0) \in M$ such that each mutual distance is an integer. Then there exists a rotation T such that $TM \subset \mathbf{Z}^3$.*

By the same argument showing Theorem 1 to be equivalent to Theorem 2, Theorem 6 is equivalent to the following.

Theorem 7. *Let p be a prime. Let $M \subset \mathbf{Z}^3$ be a finite set of points with $(0,0,0) \in M$ such that each mutual distance is an integer divisible by p. Then there exists a rotation T such that $TM \subset p\mathbf{Z}^3$.*

To prove Theorem 7, we will use the arithmetic of Lipschitz-integral quaternions and their interpretation as three-dimensional rotations. A quaternion $\mathbf{t} = t_0 + t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$ is *Lipschitz-integral* when $t_i \in \mathbf{Z}$. The Lipschitz-integral quaternions compose a noncommutative ring L with 8 units: $\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. Given $\mathbf{t}$, the *conjugate* is $\overline{\mathbf{t}} = t_0 - t_1\mathbf{i} - t_2\mathbf{j} - t_3\mathbf{k}$, the *norm* is $\mathrm{N}(\mathbf{t}) = \overline{\mathbf{t}}\mathbf{t} = t_0^2 + t_1^2 + t_2^2 + t_3^2$, and $\mathbf{t}$ is *pure* when $t_0 = 0$, i.e., $\overline{\mathbf{t}} = -\mathbf{t}$. Under the identification $(x, y, z) \leftrightarrow x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, points with integer coordinates are identified with pure Lipschitz-integral quaternions. We observe again that statements about distances can be translated into statements about norms. Each (not necessarily pure) Lipschitz-integral quaternion $\mathbf{t}$ gives rise to a rotation operator that takes pure quaternions to pure quaternions:

$$T_{\mathbf{t}} \colon (x, y, z) \mapsto \tfrac{1}{\mathrm{N}(\mathbf{t})}\mathbf{t}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\overline{\mathbf{t}}. \tag{2}$$

Compared to the ring $\mathbf{Z}[\mathbf{i}]$ of Gaussian integers, where only *certain* primes $p \in \mathbf{Z}$ occur as a norm but then the factorization $p = \overline{\pi}\pi$ is essentially unique, the arithmetic in L is unusual. *Each* prime $p \in \mathbf{Z}$ occurs as a norm (as a consequence of Lagrange's four-square theorem), but the factorization $p = \overline{\mathbf{t}}\mathbf{t}$ is never unique (not even up to left unit multiplication, right unit multiplication, or conjugates); for example, consider $5 = (2+\mathbf{i})(2-\mathbf{i}) = (2+\mathbf{j})(2-\mathbf{j})$. Jacobi's four-square theorem tells us that, for each prime $p \in \mathbf{Z}$, there are $p + 1$ primes in L of norm p (up to multiplication by units). Fortunately, the arithmetic in L turns out to be sufficiently reasonable for our needs.

Lemma 8. *Given* $\mathbf{u} \in L$*, if* p *is an odd prime such that* $p \nmid \mathbf{u}$ *but* $p \mid \mathrm{N}(\mathbf{u})$*, then* $\mathbf{u}$ *has a unique right divisor of norm* p *(unique up to left unit multiplication), and* $\mathbf{u}$ *has a unique left divisor of norm* p *(unique up to right unit multiplication).*

Proof. See [6, Theorem 1]. □

Lemma 9. *Given* $\mathbf{u}, \mathbf{u}' \in L$*, if* p *is an odd prime such that* p *divides neither* $\mathbf{u}$ *nor* $\mathbf{u}'$*, but* p *divides both* $\mathrm{N}(\mathbf{u})$ *and* $\mathrm{N}(\mathbf{u}')$*, then* $\mathbf{u}$ *and* $\mathbf{u}'$ *share a divisor of norm* p *(either the same left divisor, or the same right divisor, or both) if and only if* $p \mid \mathrm{N}(\mathbf{u} - \mathbf{u}')$*.*

Proof. See [6, Theorem 7]. □

Remark. As in the two-dimensional method, we do not need Lemma 8 and Lemma 9 for the case $p = 2$, but unlike the two-dimensional method, where the corresponding lemmata held for $p = 2$, here they do not. For example, $1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$ admits both $1 + \mathbf{i}$ and $1 + \mathbf{j}$ as right divisors, and $1 + \mathbf{i}$ and $1 + \mathbf{j}$ do *not* share a divisor of norm 2 yet $2 \mid \mathrm{N}(\mathbf{i} - \mathbf{j})$.

Lemma 10. *Let* p *be a prime. If* $\mathbf{u} \in L$ *is pure and satisfies* $p \nmid \mathbf{u}$ *but* $p^2 \mid \mathrm{N}(\mathbf{u})$*, then* $\mathbf{u}$ *has a factorization of the form* $\mathbf{u} = \bar{\mathbf{t}}\mathbf{s}\mathbf{t}$ *for a unique* $\mathbf{t}$ *of norm* p *(unique up to left unit multiplication). Furthermore, if* $\mathbf{u}' \in L$ *satisfies the same hypotheses as* $\mathbf{u}$*, then* $\mathbf{u}'$ *has a factorization* $\mathbf{u}' = \bar{\mathbf{t}}\mathbf{s}'\mathbf{t}$ *(where* $\mathbf{t}$ *is the same as for* $\mathbf{u}$*) if and only if* $p \mid \mathrm{N}(\mathbf{u} - \mathbf{u}')$*.*

Proof. We must have p odd since $2 \nmid \mathbf{u}$ is incompatible with $\mathrm{N}(\mathbf{u}) \equiv 0 \pmod 4$. By Lemma 8, $\mathbf{u}$ has unique left and right divisors of norm p. Let $\mathbf{t}$ be the right divisor. Conjugating $\mathbf{u} = \mathbf{w}\mathbf{t}$ gives $\mathbf{u} = \bar{\mathbf{t}}(-\overline{\mathbf{w}})$, so $\bar{\mathbf{t}}$ must be the left divisor. Now $p \nmid \mathbf{w}$ but $p \mid \mathrm{N}(\mathbf{w})$. By Lemma 8, $\mathbf{w}$ has a unique left divisor of norm p, which is then a left divisor of $\mathbf{u}$, thus must be $\bar{\mathbf{t}}$. For the second part, apply Lemma 9. □

We can now prove Theorem 7.

Proof of Theorem 7. Using the identification $(x, y, z) \leftrightarrow x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we get that M is a set of pure Lipschitz-integral quaternions satisfying

$$p^2 \mid \mathrm{N}(\mathbf{u}) \quad \text{and} \quad p^2 \mid \mathrm{N}(\mathbf{u} - \mathbf{u}'), \quad \text{for all } \mathbf{u}, \mathbf{u}' \in M.$$

If $p \mid \mathbf{u}$ for all $\mathbf{u} \in M$, then already $M \subset p\mathbf{Z}^3$. Otherwise, there exists $\mathbf{u}$ with $p \nmid \mathbf{u}$. By Lemma 10, there is a factorization $\mathbf{u} = \bar{\mathbf{t}}\mathbf{s}\mathbf{t}$ where $\mathbf{t}$ has norm p. For any other $\mathbf{u}' \in M$, if $p \nmid \mathbf{u}'$, then Lemma 10 tells us that $\mathbf{u}' = \bar{\mathbf{t}}\mathbf{s}'\mathbf{t}$, while if $p \mid \mathbf{u}'$, then $p \nmid \mathbf{u} - \mathbf{u}'$, so Lemma 10 tells us that $\mathbf{u} - \mathbf{u}' = \bar{\mathbf{t}}\mathbf{s}'\mathbf{t}$, whence $\mathbf{u}' = \bar{\mathbf{t}}(\mathbf{s} - \mathbf{s}')\mathbf{t}$. Either way, every $\mathbf{v} \in M$ factors in the form $\mathbf{v} = \bar{\mathbf{t}}\mathbf{s}_{\mathbf{v}}\mathbf{t}$, whence $p^2 \mid \mathbf{t}\mathbf{v}\bar{\mathbf{t}}$; therefore, $T_{\mathbf{t}}M \subset p\mathbf{Z}^3$, where $T_{\mathbf{t}}$ is the rotation operator (2). □

Remark. During preparation for publication, we learned of the work of W. F. Lunnon [5] on the same topic. Lunnon pointed out to us (private communication, 2012) that Fricke's method goes through almost unchanged when the definition of integral distance sets is relaxed to require only the *square* of the mutual distances to be integers. Indeed, with that relaxed definition, only minor changes are required in the statements of Theorem 2 and Theorem 7 (and the proofs of equivalence to Theorem 1 and Theorem 6), while the preparatory lemmata and actual proofs of Theorem 2 and Theorem 7 go through unchanged. We kept Fricke's original definition for simplicity of presentation.

5 Tetrahedron example.

We wish to realize the Heronian tetrahedron $(612, 480, 156, 185, 319, 455)$, which has volume 665280, and whose face $(612, 480, 156)$ has area 22464, as a lattice tetrahedron. Under the identification $(x, y, z) \leftrightarrow x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we position the

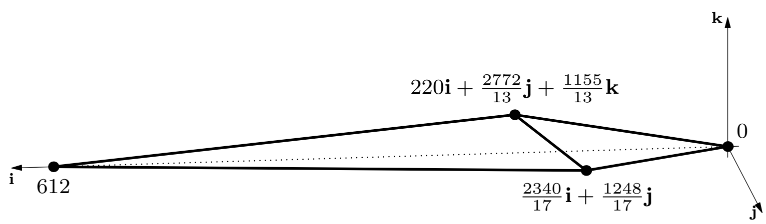


Figure 8: Tetrahedron $(612, 480, 156, 185, 319, 455)$ positioned rationally

tetrahedron initially with rational coordinates, as shown in Figure 8. To clear denominators, we scale up by $13 \cdot 17$ to obtain the vertices

$$0, \quad A = 135252\mathbf{i}, \quad B = 30420\mathbf{i} + 16224\mathbf{j}, \quad \text{and} \quad C = 48620\mathbf{i} + 47124\mathbf{j} + 19635\mathbf{k}.$$

The scaled vertices satisfy the hypotheses of Theorem 7 for $p = 13$ and $p = 17$.

We consider $p = 13$ and observe $13 \nmid C$ but $13 \mid \mathrm{N}(C)$. Thus, we seek the unique quaternion $\mathbf{t}$ of norm 13 which divides C on the right, whose existence is guaranteed by Lemma 8. We note that the divisibility condition is equivalent to $13 \mid C\overline{\mathbf{t}}$. There are 14 quaternions of norm 13 (up to multiplication by units), which we find by expressing 13 as a sum of four squares in all possible ways as $13 = 2^2 + 3^2$ and $13 = 1^2 + 2^2 + 2^2 + 2^2$. The first expression leads to 6 quaternions of norm 13: $2 \pm 3\mathbf{i}$, $2 \pm 3\mathbf{j}$, and $2 \pm 3\mathbf{k}$. The second expression leads to 8 quaternions of norm 13: $1 \pm 2\mathbf{i} \pm 2\mathbf{j} \pm 2\mathbf{k}$. We conclude $\mathbf{t} = 2 - 3\mathbf{i}$ by verifying

$$C \cdot \overline{2 - 3\mathbf{i}} = 13 \cdot (-11220 + 7480\mathbf{i} + 11781\mathbf{j} - 7854\mathbf{k}).$$

The proof of Theorem 7 tells us that all vertices have the form $(\overline{2 - 3\mathbf{i}})\mathbf{s}(2 - 3\mathbf{i})$, allowing us to rotate by $T_{2-3\mathbf{i}}$; that is, we multiply all coordinates by $2 + 3\mathbf{i}$ on the right, by $2 - 3\mathbf{i}$ on the left, and then divide by 13. The result is a new position in which all coordinates are divisible by 13, allowing us to scale back down by 13, and we obtain the vertices

$$0, \quad A' = 10404\mathbf{i}, \quad B' = 2340\mathbf{i} - 480\mathbf{j} - 1152\mathbf{k}, \quad \text{and} \quad C' = 3740\mathbf{i} - 3927\mathbf{k}.$$

We now consider $p = 17$. The vertices still satisfy Theorem 7 for $p = 17$ and we observe $17 \nmid B'$. Thus, we seek the unique quaternion $\mathbf{t}$ of norm 17 which divides B' on the right. There are 18 quaternions of norm 17, which we find by writing 17 as a sum of 4 squares in all possible ways as $1^2 + 4^2$ (this leads to 6 quaternions of norm 17) and $3^2 + 2^2 + 2^2$ (this leads to 12 quaternions of norm 17). The quaternion we seek is $\mathbf{t} = 3 - 2\mathbf{j} - 2\mathbf{k}$:

$$B' \cdot \overline{3 - 2\mathbf{j} - 2\mathbf{k}} = 17 \cdot (192 + 492\mathbf{i} - 360\mathbf{j} + 72\mathbf{k}).$$

After rotating by $T_{3-2\mathbf{j}-2\mathbf{k}}$ and then scaling down by 17, we obtain the vertices

$$0, \quad 36\mathbf{i} - 432\mathbf{j} + 432\mathbf{k}, \quad 36\mathbf{i} - 144\mathbf{j} + 48\mathbf{k}, \quad \text{and} \quad 176\mathbf{i} - 264\mathbf{j} + 33\mathbf{k}.$$

The position as a lattice tetrahedron is shown in Figure 9.

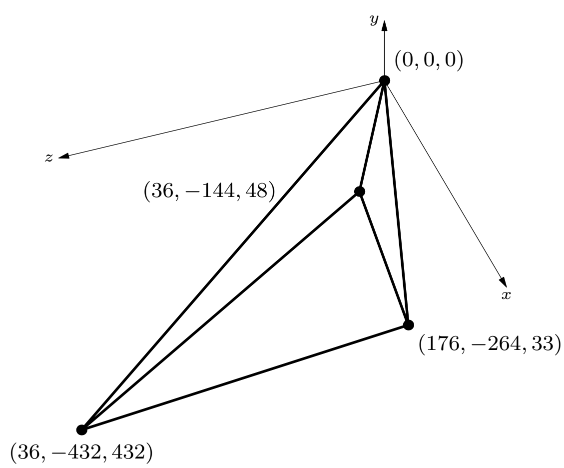


Figure 9: Tetrahedron $(612, 480, 156, 185, 319, 455)$ as a lattice tetrahedron

6 Recovering Fricke's method from the three-dimensional case.

Since the two-dimensional approach for triangles using Gaussian integers is almost statement-for-statement analogous to the three-dimensional approach for tetrahedra using Lipschitz-integral quaternions, it seems that the former ought to be a special case of the latter. This is indeed true.

Theorem 11. *In the proof of Theorem 7, if each* $\mathbf{u} \in M \subset \mathbf{Z}^3$ *has* $z = 0$*, i.e.,* $M \subset \mathbf{Z}^2$*, then among the left associate choices of rotation operator* $T_{\mathbf{t}}$*, there exists one that keeps* M *in the xy-plane, i.e., has the z-axis as its axis of rotation.*

Proof. For $\mathbf{t} = t_0 + (t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k})$, the axis of rotation of $T_{\mathbf{t}}$ is identified by the vector (t_1, t_2, t_3). Thus $T_{\mathbf{t}}$ has the z-axis as its axis of rotation precisely when $\mathbf{t}$ has the form $\mathbf{t} = t_0 + t_3\mathbf{k}$. Tracing the proof of Theorem 7 back through Lemma 10 to Lemma 8, we must therefore show the following. *If* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ *is a pure Lipschitz-integral quaternion, representing a point in the xy-plane, with* $p \nmid \mathbf{u}$ *but* $p \mid \mathrm{N}(\mathbf{u})$ *for an odd prime* p*, then* $\mathbf{u}$ *admits a right divisor* $\mathbf{t}$ *of norm* p *of the form* $\mathbf{t} = t_0 + t_3\mathbf{k}$*.*

The equation $\mathbf{u} = \mathbf{v}\mathbf{t}$ is equivalent to $\mathbf{u}\bar{\mathbf{t}} \equiv 0 \pmod{p}$. By assumption, $u_1^2 \equiv -u_2^2$, and $p \nmid \mathbf{u}$ tells us that $u_1 \not\equiv 0$, so -1 is a square modulo p. Let $s^2 \equiv -1$, where we choose the sign on s so that $u_1 \equiv su_2$. This also means $p \equiv 1 \pmod{4}$, so there exist a, b with $p = a^2 + b^2$, and here we choose signs so that $a \equiv -sb$. Let $\mathbf{t} = a + b\mathbf{k}$. Then $\mathrm{N}(\mathbf{t}) = p$ and $\mathbf{u}\bar{\mathbf{t}} \equiv 0$. □

To recover the Gaussian version of Fricke's method, we identify $\mathbf{t} = t_0 + t_3\mathbf{k}$ with the Gaussian integer $\pi = t_0 + t_3\mathbf{i}$, and identify points $(x, y, 0)$ with $x + y\mathbf{i}$. Then the quaternion rotation operator $T_{\mathbf{t}}$ morphs into the Gaussian rotation operator T_π, Lemmata 8–10 respectively morph into Lemmata 3–5, and the statement and proof of Theorem 7 morph into the statement and proof of Theorem 2.

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