Instructions. Answer each of the questions on your own paper and put your name on each page of your paper.

The following definitions are provided for your convenience: If R is an integral domain, recall that an element a of R is a *unit* if $ab = 1$ for some $b \in R$; an element a is *prime* if a is not a unit and whenever $a|bc$ then $a|b$ or $a|c$; and an element a is *irreducible* if a is not a unit and whenever $a = bc$ then either b or c is a unit.

1. (a) Define the terms *prime ideal* and *maximal ideal* in a commutative ring R with identity.

   ▶ **Solution.** See Definitions 2.15 and 2.19, Pages 62, 63 of the text. ◀

   (b) Let R be a commutative ring with identity. If M is a maximal ideal of R, prove that M is a prime ideal.

   ▶ **Solution.** This is Theorem 2.21, Page 63 of the text. Two proofs are given there. Here is yet another. Suppose that $ab \in M$ but $a \notin M$. Then the ideal $M + \langle a \rangle$ properly contains M since $a \notin M$. Thus $M + \langle a \rangle = R$ and we have an equation $m + ac = 1$, where $m \in M$ and $c \in R$. Multiply by b to get $b = bm + abc \in M$ since $m \in M$ and $ab \in M$. Thus, M is prime. ◀

   (c) For the polynomial ring $R = \mathbb{Z}[X]$, show that the principal ideal $P = \langle X \rangle$ is an example of a nonzero prime ideal P of R such that P is *not* maximal.

   ▶ **Solution.** Let $\varphi : \mathbb{Z}[X] \to \mathbb{Z}$ by $\varphi(f(X)) = f(0)$. Then φ is a surjective ring homomorphism with $\text{Ker}(\varphi) = \langle X \rangle$. Thus $\mathbb{Z}[X]/\langle X \rangle \cong \mathbb{Z}$, which is an integral domain but not a field. Thus, $\langle X \rangle$ is prime but not maximal. ◀

2. Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$.

   (a) Why is R an integral domain?

   ▶ **Solution.** R is a subring of an integral domain, namely $\mathbb{C}$, and hence an integral domain. ◀

   (b) What are the units in R?

   ▶ **Solution.** Suppose $a + b\sqrt{-5}$ is a unit of R. Then there is an equation in R: $1 = (a + b\sqrt{-5})(c + d\sqrt{-5})$. Multiplying by the complex conjugates gives an equation $1 = (a^2 + 5b^2)(c^2 + 5d^2)$ in integers. Thus we must have $a^2 + 5b^2 = 1$, which can only happen if $b = 0$. Hence $a^2 = 1$ so $a = \pm 1$ and $R^* = \{\pm 1\}$. ◀

   (c) Is the element 2 irreducible in R?

   ▶ **Solution.** Suppose we can write $2 = (a + b\sqrt{-5})(c + d\sqrt{-5})$ in R. Multiplying by the complex conjugates gives an equation in integers

   $$(*) \qquad 4 = (a^2 + 5b^2)(c^2 + 5d^2).$$

   Thus, each integer on the right is either 1, 2, or 4. But the equation $x^2 + 5y^2 = 2$ has no solutions in integers since $y \neq 0 \implies x^2 + 5y^2 \geq 5$, so we must have $y = 0$. But that would require $x^2 = 2$, which has no solutions in integers. Hence, one of the two factors on the right is 1 and the other is 4. So either $a^2+5b^2=1$ or $c^2+5d^2=1$, which gives, as in the previous part, that $a+b\sqrt{-5}=\pm 1$ or $c+d\sqrt{-5}=\pm 1$. That is, if 2 factors in R then one of the factors is a unit, so that 2 is irreducible in R ◀

(d) Is 2 a prime element of R?

▶ **Solution.** No. 2 is not prime since $2=2\cdot 3=(1+\sqrt{-5})(1-\sqrt{-5})$, so 2 divides the product $(1+\sqrt{-5})(1-\sqrt{-5})$, but 2 does not divide $1\pm\sqrt{-5}$ in R. ◀

3. Let $\mathbb{F}$ be a field and let R be the following subring of the ring of 2×2 matrices $M_2(\mathbb{F})$:

$$R=\left\{\begin{bmatrix} a & b\\ 0 & c\end{bmatrix} : a,\, b,\, c, \in \mathbb{F}\right\},$$

and let $J=\left\{\begin{bmatrix} 0 & d\\ 0 & 0\end{bmatrix} : d\in\mathbb{F}\right\}\subseteq R$. Show that J is a (two-sided) ideal of R and that there is a ring isomorphism

$$\varphi : R/J\to D,$$

where,

$$D=\left\{\begin{bmatrix} a & 0\\ 0 & c\end{bmatrix} : a,\, c, \in \mathbb{F}\right\},$$

is the subring of $M_2(\mathbb{F})$ consisting of diagonal matrices.

▶ **Solution.** Define $f: R\to D$ by

$$f\left(\begin{bmatrix} a & b\\ 0 & c\end{bmatrix}\right)=\begin{bmatrix} a & 0\\ 0 & c\end{bmatrix}.$$

Since

$$\begin{aligned} f\left(\begin{bmatrix} a & b\\ 0 & c\end{bmatrix}\begin{bmatrix} a' & b'\\ 0 & c'\end{bmatrix}\right) &= f\left(\begin{bmatrix} aa' & ab'+bc'\\ 0 & cc'\end{bmatrix}\right)=\begin{bmatrix} aa' & 0\\ 0 & cc'\end{bmatrix}\\ &= \begin{bmatrix} a & 0\\ 0 & c\end{bmatrix}\begin{bmatrix} a' & 0\\ 0 & c'\end{bmatrix}=f\left(\begin{bmatrix} a & b\\ 0 & c\end{bmatrix}\right)f\left(\begin{bmatrix} a' & b'\\ 0 & c'\end{bmatrix}\right).\end{aligned}$$

Thus f preserves multiplication and preservation of addition is similar. Hence f is a ring homomorphism with $\mathrm{Ker}(f)=J$. Moreover, f is surjective so the first isomorphism theorem gives a ring isomorphism

$$\varphi=\overline{f}: R/J\to D.$$

◀

4. Recall that if V is a vector space over a field $\mathbb{F}$ and $T: V\to V$ is a linear transformation, then the vector space V is made into an $\mathbb{F}[X]$ module V_T by defining the scalar multiplication

$$f(X)v=f(T)(v).$$

For this exercise, we will let $\mathbb{F}=\mathbb{R}$, $V=\mathbb{R}^2$ and $T:\mathbb{R}^2\to\mathbb{R}^2$ will be the $\mathbb{R}$-linear transformation defined by the formula

$$T(x_1,\, x_2)=(2x_1,\, 3x_2).$$

Answer the following questions concerning the $\mathbb{R}[X]$ module V_T determined by this linear transformation T. Let $v=(x_1,\, x_2)$.

(a) Find all v such that $Xv = 2v$ and find all v such that $Xv = 3v$.

▶ **Solution.** $Xv = 2v$ if and only if $(2x_1, 3x_2) = (2x_1, 2x_2)$ which is true if and only if $x_2 = 0$. Thus, $Xv = 2v \iff v = (x_1, 0)$, and similarly, $Xv = 3v \iff v = (0, x_2)$. ◀

(b) Show $Xv = cv$ for some $c \in \mathbb{R}$ and $v \neq 0$ if and only if $c \in \{2, 3\}$.

▶ **Solution.** If $v = (x_1, x_2)$, then $Xv = cv \iff 2x_1 = cx_1$ and $3x_2 = cx_2$. If $x_1 \neq 0$ then this forces $c = 2$, while $x_2 \neq 0$ forces $c = 3$. ◀

(c) Compute $(X^2 - 5X + 6)v$.

▶ **Solution.** If $v = (x_1, x_2)$, then $(X^2 - 5X + 6)v = (X - 3)(X - 2)v = (X - 3)(0, x_2) = (0, 0)$. ◀

(d) There are exactly two $\mathbb{R}[X]$-submodules of V in addition to $\{0\}$ and V. Find them.

▶ **Solution.** By part (b), the only T-invariant subspaces of V (other than $\{0\}$ and V) are the two coordinate axes $V_1 = \{(x_1, 0) : x_1 \in \mathbb{R}\}$ and $V_2 = \{(0, x_2) : x_2 \in \mathbb{R}\}$. Thus, V_1 and V_2 are the only two $\mathbb{R}[X]$-submodules of V_T other than $\{0\}$ and V. ◀