Do the following exercises from the text:
Section 8.1: 2

2. Let $p(n)$ denote the number of distinct positive divisors of n. Let $q(n) = a^{p(n)}$ where a is fixed and show that $q(n)$ is multiplicative, but not completely multiplicative.

▶ **Solution.** Suppose that $(m, n) = 1$. Then $m = \prod_{i=1}^{r} p_i^{k_i}$ and $n = \prod_{i=r+1}^{r+s} p_i^{k_i}$ where $k_i \geq 1$ and $p_1, p_2, \ldots, p_{r+s}$ are distinct primes. Then $mn = \prod_{i=1}^{r+s} p_i^{k_i}$. Therefore

$$q(mn) = a^{p(mn)} = a^{r+s} = a^r a^s = a^{p(m)} a^{p(n)} = q(m)q(n)$$

and hence q is multiplicative. However, $q(6) = a^2$, $q(4) = a$ and $q(6 \cdot 4) = a^2 \neq q(6)q(4)$. Thus, q is not completely multiplicative. ◀

Section 8.2: 3

3. Show that

$$\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}$$

for every positive integer n.

▶ **Solution.** Since n/d runs through the divisors of n (backwards) as d runs through the divisors (forward), it follows that

$$\sum_{d|n} \frac{1}{d} = \sum_{d|n} \frac{1}{n/d} = \sum_{d|d} \frac{d}{n} = \frac{1}{n} \sum_{d|n} d = \frac{\sigma(n)}{n}.$$

◀

Section 8.3: 3

3. Show that $\sigma_2(n) = \sigma(n) \cdot \displaystyle\prod_{i=1}^{r} \frac{p_i^{n_i+1}+1}{p_i+1}$, where $n = \prod_{i=1}^{r} p_i^{n_i}$ is the canonical representation of n.

▶ **Solution.** By Theorem 8.6,

$$\begin{aligned}
\sigma_2(n) &= \prod_{i=1}^{r} \frac{p_i^{2(n_i+1)-1}}{p_i^2 - 1} \\
&= \prod_{i=1}^{r} \frac{(p_i^{n_1+1} - 1)(p_i^{n_i+1} + 1)}{(p_i - 1)(p_i + 1)} \\
&= \prod_{i=1}^{r} \frac{p_i^{n_i+1} - 1}{p_i - 1} \cdot \prod_{i=1}^{r} \frac{p_i^{n_i+1} + 1}{p_i + 1} = \sigma(n) \prod_{i=1}^{r} \frac{p_i^{n_i+1} + 1}{p_i + 1}.
\end{aligned}$$

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Additional Exercises on the Möbius function (Section 8.4).

1. Find the following values of the Möbius function.

   (a) $\mu(12)$ (b) $\mu(15)$ (c) $\mu(30)$ (d) $\mu(50)$

   (e) $\mu(1001)$ (f) $\mu(2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13)$ (g) $\mu(10!)$

   ▶ **Solution.** (a) $\mu(12) = 0$, (b) $\mu(15) = 1$, (C) $\mu(30) = -1$, (d) $\mu(50 = 0$, (e) $\mu(1001) = \mu(7\cdot 11\cdot 13) = -1$, (f)$\mu(2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13) = (-1)^6 = 1$, (g) $\mu(10!) = 0$ since $4 \mid 10!$. ◀

2. Show that if n is a positive integer, then $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$.

   ▶ **Solution.** One of the 4 consecutive numbers n, $n+1$, $n+2$, $n+3$ is divisible by 4 and $\mu(m) = 0$ whenever $4 \mid m$. Hence the product $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$. ◀

3. Suppose that f is a multiplicative function with $f(1) = 1$. Show that

   $$\sum_{d\mid n} \mu(d)f(d) = (1 - f(p_1))(1 - f(p_2))\cdots(1 - f(p_t)),$$

   where $p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t}$ is the prime power factorization of n.

   ▶ **Solution.** Since f is multiplicative and μ is multiplicative, the product $f\mu$ is multiplicative and thus, the divisor sum function $g(n) = \sum_{d\mid n} \mu(d)f(d)$ is also multiplicative, and thus it can be evaluated by computing the value for powers of a prime. Then, if $n = p^k$,

   $$g(p^k) = \sum_{d\mid p^k} \mu(d)f(d) = \sum_{j=0}^{k} \mu(p^j)f(p^j) = \mu(1)f(1) + \mu(p)f(p) = 1 - f(p)$$

   since $\mu(p^j) = 0$ for $j \geq 2$. The formula now follows from the fact that $g(n)$ is multiplicative. ◀