Do the following exercises from the text:
Section 1.7: 6, 8

6. If a is an integer, prove that one of the numbers a, $a+2$, and $a+4$ is divisible by 3.

▶ **Solution.** Divide a by 3. By the division algorithm, there exist unique integers q and r with $0 \leq r < 3$ and $a = 3q + r$. If $r = 0$, then $3 \mid a$. If $r = 1$, then $a+2 = 3q+1+2 = 3q+3 = 3(q+1)$ and $3 \mid (a+2)$. If $r = 2$, then $a+4 = 3q+2+4 = 3q+6 = 3(q+2)$ and $3 \mid (a+4)$. Therefore, $3 \mid a$, or $3 \mid (a+2)$, or $3 \mid (a+4)$. ◀

8. If a, b, and c are integers with $a^2 + b^2 = c^2$, show that a and b cannot both be odd.

▶ **Solution.** If a and b are both odd, then $a = 2k+1$ and $b = 2m+1$. Hence $a^2 = 4k^2 + 4k + 1$ and $b^2 = 4m^2 + 4m + 1$ so $a^2 + b^2 = 4(k^2 + k + m^2 + m) + 2$ so that $a^2 + b^2$ must have remainder 2 when divided by 4. But if $c = 2r$ then $c^2 = 4r^2$ so c^2 has remainder 0 when divided by 4, and if $c = 2s+1$, then $c^2 = 4(s^2 + s) + 1$ so c^2 has remainder 1 when divided by 4. Therefore, the square of any integer must be 0 or 1 when divided by 4. However, if a and b are odd, then we have seen that $a^2 + b^2$ has remainder 2 when divided by 4. Thus, it is not possible for $a^2 + b^2$ to be the square of an integer if both a and b are odd. ◀

Section 2.1: 4

4. If $m \mid (8n+7)$ and $m \mid (6n+5)$, prove that $m = \pm 1$.

▶ **Solution.** Since $m \mid (8n+7)$ and $m \mid (6n+5)$, and

$$1 = 3(8n+7) - 4(6n+5),$$

it follows that $m \mid 1$. Therefore, $m = \pm 1$. ◀

Section 2.3: 2, 4, 14

2. **(a)** Compute (7700, 2233) and determine x and y such that

$$(7700,\ 2233) = 7700x + 2233y.$$

▶ **Solution.** Use the Euclidean algorithm and record the successive divisiions in the following table:

7700	2233	
1	0	7700
0	1	2233
1	−3	1001
−2	7	231
9	−31	77
−29	100	0

Thus, $(7700, 2233) = 77 = 7700 \cdot 9 + 2233(-31)$. ◀

(b) Compute $(7700,\ -2233)$ and determine x and y such that

$$(7700,\ -2233) = 7700x - 2233y.$$

▶ **Solution.** Since, $(a,\ b) = (a,\ -b)$ because the divisors of b and $-b$ are the same, it follows that

$$(7700,\ -2233) = (7700, 2233) = 77 = 7700 \cdot 9 + 2233 \cdot (-31)$$

◀

4. If $b \neq 0$ prove that $(0,\ b) = |b|$.

▶ **Solution.** Since $0 = 0 \cdot |b|$ and $b = \pm 1 \cdot |b|$, it follows that $|b|$ is a common divisor of 0 and b. Let c be any other common divisor of 0 and b. Then $b = cs$ for some integer s and then $|b| = |cs| = |c|\,|s| \geq |c|$ since s is a nonzero integer and hence $|s| \geq 1$. Thus $c \leq |c| \leq |b|$ so that $|b|$ is the largest of the common divisors of 0 and b. That is, $(0,\ b) = |b|$. ◀

14. Prove that the product of any three consecutive integers is divisible by 6.

▶ **Solution.** Consider any three consecutive integers a, $a+1$ and $a+2$, and let $m = a(a+1)(a+2)$. If a is even then $a = 2k$ and $2 \mid a$. If a is not even then $a = 2k+1$ and $a+1 = 2k+2 = 2(k+1)$ so $2 \mid (a+1)$. In either case, $2 \mid a$ or $2 \mid (a+1)$ and so $2 \mid m$. Similarly, divide a by 3 to get $a = 3q + r$. If $r = 0$, $= 3q$ so $3 \mid a$. If $r = 1$, then $a+2 = 3q+1+2 = 3(q+1)$ so $3 \mid (a+2)$. If $r = 2$ then $a+1 = 3q+1+2 = 3(q+1)$ so $3 \mid (a+1)$. So 3 divides either a, $a+1$, or $a+2$, and hence $3 \mid m$. Since $2 \mid m$ and $3 \mid m$ and $(2,\ 3) = 1$, Theorem 2.13 shows that $2 \cdot 3 = 6$ divides m. ◀

Section 2.4: 1(c), 4, 5, 8

1. (c) Find [299, 377].

▶ **Solution.** First compute $(377,\ 299)$ by the Euclidean algorithm:

377	299	
1	0	377
0	1	299
1	−1	78
−3	4	65
4	−5	13
−23	29	0

From this we conclude that $(377,\ 299) = 13$. Then

$$[377,\ 299] = \frac{377 \cdot 299}{(377,\ 299)} = \frac{377 \cdot 299}{13} = \frac{112723}{13} = 8671.$$

◀

4. Find (299, 377, 403) and x, y, and z such that

$$(299,\, 377,\, 403) = 299x + 377y + 403z.$$

▶ **Solution.** By Theorem 2.20 and exercise 1 (c), $(299, 377, 403) = ((299, 377), 403) = (13, 403) = 13$ since $403 = 13 \cdot 31$. From the Euclidean algorithm calculation done in 1 (c), $13 = 299(-5) + 377 \cdot 4$, so

$$(299,\, 377,\, 403) = 13 = 299(-5) + 377 \cdot 4 + 403 \cdot 0.$$

◀

5. Find [299, 377, 403].

▶ **Solution.** From 1 (c), $[299, 377] = 8671$. Then from Theorems 2.21 and 2.19,

$$[299,\, 377,\, 403] = [[299,\, 377],\, 403] = [8671,\, 403] = \frac{8671 \cdot 403}{(8671,\, 403)}.$$

Use the Euclidean algorithm to calculate (8671, 403):

8671	403	
1	0	8671
0	1	403
1	−21	208
−1	22	195
2	−43	13
−31	667	0

Hence $(8671, 403) = 13$ and $[299, 377, 403] = \frac{8671 \cdot 403}{13} = \frac{3494413}{13} = 268801$. ◀

8. For any integer n, prove that $[9n + 8, 6n + 5] = 54n^2 + 93n + 40$.

▶ **Solution.** Since $(9n + 8)(6n + 5) = 54n^2 + 93n + 40$, the result will follow from Theorem 2.19 if we can show that $(9n+8, 6n+5) = 1$. Since $9n+8 = (6n+5)+(3n+3)$ and $6n + 5 = 2(3n + 3) - 1$ it follows that $(9n + 8, 6n + 5) = (6n + 5, 3n + 3) = (3n + 3, -1) = 1$ since the only divisors of -1 are ± 1. ◀

Supplemental Problem: (a) If d and n are positive integers such that $d \mid n$, prove that $(2^d - 1) \mid (2^n - 1)$.

(*Hint:* Use the identity $x^k - 1 = (x - 1)(x^{k-1} + x^{k-2} + \cdots + x + 1)$.)

▶ **Solution.** By assumption, $d \mid n$ so $n = dk$ for some positive integer k. Substitute $x = 2^d$ into the identity

$$x^k - 1 = (x - 1)(x^{k-1} + x^{k-2} + \cdots + x + 1)$$

to get

$$2^n - 1 = 2^{kd} - 1 = (2^d)^k - 1 = (2^d - 1)((2^d)^{k-1} + (2^d)^{k-2} + \cdots + (2^d) + 1)$$

Thus $2^n - 1 = (2^d - 1)s$ where s is the integer $s = (2^d)^{k-1} + (2^d)^{k-2} + \cdots + (2^d) + 1$. Hence, $(2^d - 1) \mid (2^n - 1)$. ◀

(b) Verify that $2^{35} - 1$ is divisible by 31 and 127.

▶ **Solution.** Since $35 = 7 \cdot 5$, part (a) shows that $2^{35} - 1$ is divisible by both $2^7 - 1 = 127$ and $2^5 - 1 = 31$. ◀