Verify the following results using some version of induction. Write your arguments out completely, being sure to identify the statement $P(n)$ appropriately (or the subset S of the positive integers that you will be showing is all of the positive integers).

1. Show that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all integers $n \geq 1$.

   ▶ **Solution.** The result is proved by induction. Let S be the subset of positive integers n such that
$$1 + 3 + 5 + \cdots + (2n - 1) = n^2. \qquad (*)$$
   We will use induction to show that $S = \{n \in \mathbb{Z} : n \geq 1\}$.

   Base Step. If $n = 1$, the left hand side of equation $(*)$ is 1, while the right hand side is $1^2 = 1$. Thus $1 \in S$.

   Induction Step. Suppose that $k \in S$ for a fixed but arbitrary positive integer k. This means that
$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$
   Now consider the left hand side of equation $(*)$ for $n = k + 1$. Then,
$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) &= (1 + 3 + 5 + \cdots + (2k - 1)) + (2(k + 1) - 1 \\ &= k^2 + (2(k + 1) - 1) = k^2 + 2k + 2 - 1 \\ &= (k + 1)^2, \end{aligned}$$
   where the second equality uses the induction hypothesis $k \in S$. This equation then says that if $k \in S$, then so is $k + 1 \in S$. Therefore, by the principle of induction, it follows that S consists of all positive integers, and hence, equation (*) is true for all integers $n \geq 1$. ◀

2. Show that $2^{2n-1} + 1$ is divisible by 3 for all $n \geq 1$.

   ▶ **Solution.** The proof is by induction. For an integer $n \geq 1$, let $P(n)$ be the statement "$2^{2n-1} + 1$ is divisible by 3".

   Base Step. The statement $P(1)$ is the statement "$2^{2\cdot 1-1} + 1$ is divisible by 3". But $2^{2\cdot 1-1} + 1 = 3$, which is divisible by 3. Thus, $P(1)$ is true.

   Induction Step. Now suppose that $P(k)$ is true for some k. This means that $2^{2k-1}+1$ is divisible by 3. Thus $2^{2n-1} + 1 = 3q$ for some integer q. Then
$$\begin{aligned} 2^{2(k+1)-1} + 1 &= 2^{2k-1+2} + 1 \\ &= 2^{2k-1}2^2 + 1 = 2^{2k-1}2^2 + 1 + 3 - 3 \\ &= 2^{2k-1}2^2 + 2^2 - 3 = 2^2(2^{2k-1} + 1) - 3 \\ &= 2^2(3q) - 3 = 3(2^2q - 1). \end{aligned}$$
   Since $2^2q - 1$ is an integer, this shows that $2^{2(k+1)-1} + 1$ is a multiple of 3 whenever $2^{2k-1} + 1$ is a multiple of 3. Therefore, $P(k + 1)$ is true whenever $P(k)$ is true, and the principle of induction then shows that $P(n)$ is true for all $n \geq 1$. ◀

3. Show that $f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$ for all $n \geq 1$, where f_n denotes the n^{th} Fibonacci number.

▶ **Solution.** Let S be the subset of positive integers such that

$$f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1. \tag{*}$$

We will prove by induction that S consists of all integers $n \geq 1$.

Base Step. For $n = 1$, equation (*) is $f_2 = f_3 - 1$. Since $f_2 = 1$ and $f_3 = 2$, this is a true statement $1 = 2 - 1$, so that $1 \in S$.

Induction Step. Now assume that $k \in S$ for some $k \geq 1$. This means that equation (*) is true for $n = k$, so that $f_2 + f_4 + \cdots + f_{2k} = f_{2k+1} - 1$. Then,

$$\begin{aligned} f_2 + f_4 + \cdots + f_{2k} + f_{2(k+1)} &= (f_2 + f_4 + \cdots + f_{2k}) + f_{2(k+1)} \\ &= (f_{2k+1} - 1) + f_{2k+2} = (f_{2k+1} + f_{2k+2}) - 1 \\ &= f_{2k+3} - 1 \\ &= f_{2(k+1)+1} - 1, \end{aligned}$$

where the second equality uses the induction hypothesis $k \in S$. Thus, if $k \in S$, it follows that $k + 1 \in S$ and by the principle of induction, S consists of all integers $n \geq 1$, so equation (*) is true for all $n \geq 1$. ◀

4. Show that for all $n \geq 1$,

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}.$$

▶ **Solution.** Let $P(n)$ be the statement

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}. \tag{*}$$

We will prove that this statement is true for all $n \geq 1$ by induction.

Base Step. If $n = 1$ then equation (*) becomes $\frac{1}{2^1} = 2 - \frac{1+2}{2^1}$. This is just $\frac{1}{2} = \frac{3}{2}$ which is a true statement.

Induction Step. Now assume that for a particular k that $P(k)$ is true. That is, assume that

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}.$$

Then for $n = k + 1$

$$\begin{aligned}\sum_{j=1}^{k+1} \frac{j}{2^j} &= \sum_{j=1}^{k} \frac{j}{2^j} + \frac{k+1}{2^{k+1}} \\ &= \left(2 - \frac{k+2}{2^k}\right) + \frac{k+1}{2^{k+1}} \\ &= 2 - \frac{2k+4}{2^{k+1}} + \frac{k+1}{2^{k+1}} \\ &= 2 - \frac{2k+4-(k+1)}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}} \\ &= 2 - \frac{(k+1)+2}{2^{k+1}}.\end{aligned}$$

Thus, if $P(k)$ is true for k, it is also true for $k + 1$ and by the principle of induction, $P(n)$ is true for all $n \geq 1$. ◀

5. Show that $2^n < n!$ for all $n \geq 4$. Recall that for a positive integer n, $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.

► **Solution.** Let $P(n)$ be the statement "$2^n < n!$". We show by induction that $P(n)$ is true for all $n \geq 4$.

Base Step. $2^4 = 16 < 24 = 4!$. Thus $P(4)$ is true.

Induction Step. Now assume that for a particular $k \geq 4$ that $P(k)$ is true. That is, assume that $2^k < k!$. Then

$$2^{k+1} = 2^k \cdot 2 < 2 \cdot k! < (k+1)k! = (k+1)!,$$

where the first $<$ is the induction hypothesis, and the second $<$ is because $2 < k + 1$ since $k \geq 4$.

Thus, if $P(k)$ is true for some $k \geq 4$, then so is $P(k + 1)$. Hence, by the principle of induction, $P(n)$ is true for all $n \geq 4$, since the base case starts at $n = 4$. ◀

6. Show that any integer $n \geq 12$ can be written as a sum $4r + 5s$ for some nonnegative integers r, s. (This problem is sometimes called a postage stamp problem. It says that any postage greater than 11 cents can be formed using 4 cent and 5 cent stamps.)

► **Solution.** Let $P(n)$ be the statement: "$n = 4r + 5s$ for some nonnegative integers r and s." We prove by induction that $P(n)$ is true for all $n \geq 12$.

Base Step. $12 = 4 \cdot 3 + 5 \cdot 0$ so $P(12)$ is true, with $r = 3$ and $s = 0$.

Induction Step. Now assume that for a particular $k \geq 12$ that $P(k)$ is true. That is, assume that $k = 4r+5s$ for some nonnegative integers r and s. Then $k+1 = 4r+5s+1$. Consider two cases:

Case 1: $s > 0$.

In this case $k+1 = 4r+5s+1 = 4r+5s+5-4 = 4(r-1)+5(s+1) = 4r'+5s'$ where $r' = r-1 \geq 0$ since $r > 0$ and $s' = s+1 > 0$ since $s \geq 0$.

Case 2: $s = 0$.

In this case $k = 5s$ and since $k \geq 12$, we must have $s \geq 3$. Then,

$$k+1 = 5s+1 = 5s+16-15 = 5s+4\cdot 4-5\cdot 3 = 4\cdot 4+5(s-3) = 4r'+5s'$$

where $r' = 4$ and $s' = s-3 \geq 0$ since $s \geq 3$.

Thus, in either case, if $k = 4r+5s$ for some integers $r \geq 0$, $s \geq 0$, then $k = 1 = 4r'+5s'$ for integers, $r' \geq 0$, $s' \geq 0$.

Thus, if $P(k)$ is true for some $k \geq 12$, then so is $P(k+1)$. Hence, by the principle of induction, $P(n)$ is true for all $n \geq 4$, since the base case starts at $n = 12$. ◀