**Instructions.** Answer each of the questions on your own paper, and be sure to show your work, including giving reasons, so that partial credit can be adequately assessed. Put your name on each page of your paper. There is a total possible of 100 points.

1. **[18 Points]**

   (a) Find the greatest common divisor $d = (4307, 1121)$ using the Euclidean Algorithm, and write $d = (4307, 1121)$ in the form $d = 4307 \cdot s + 1121 \cdot t$.

   ▶ **Solution.** Use the Euclidean Algorithm:

   | 4307 | 1121 | | |
   |---|---|---|---|
   | 1 | 0 | 4307 | |
   | 0 | 1 | 1121 | |
   | 1 | $-3$ | 944 | $= 4307 - 3 \cdot 1121$ |
   | $-1$ | 4 | 177 | $= 1121 - 1 \cdot 944$ |
   | 6 | $-23$ | 59 | $= 944 - 5 \cdot 177$ |
   | $-19$ | 73 | 0 | $= 177 - 3 \cdot 59$ |

   Thus, $(4307, 1121) = 59 = 4307 \cdot 6 + 1121 \cdot (-23)$. ◀

   (b) Find integers $x$ and $y$ such that $4307x + 1121y = 236$ or explain why it is not possible.

   ▶ **Solution.** Since $236 = 4 \cdot 59$ it follows from part (a) that

   $$236 = 4 \cdot 59 = 4307 \cdot 24 + 1121 \cdot (-92).$$

   ◀

   (c) Compute the least common multiple $m = [4307, 1121]$.

   ▶ **Solution.** $[4307, 1121] = \frac{4307 \cdot 1121}{59} = \frac{4828147}{59} = 81,833.$ ◀

2. **[10 Points]** Prove that $(3a + 5, 2a + 3) = 1$ for any integer $a$.

   ▶ **Solution.** Note that $2 \cdot (3a+5) - 3 \cdot (2a+3) = 10 - 9 = 1$. Thus, 1 is a linear combination of $3a+5$ and $2a+3$, and since 1 is the smallest positive integer, it follows that $(3a+5, 2a+3) = 1$ by Theorem 2.4. ◀

3. **[16 Points]** Let $a = 2^3 5^8 13^2$, $b = 2^5 3^7 5^3$. Find the following:

   (a) The prime factorization of $(a, b)$.

   (b) The prime factorization of $[a, b]$.

   (c) The largest $e$ such that $2^e \mid a^3 b$.

   (d) The largest $f$ such that $5^f \mid (a + b)$.

   ▶ **Solution.** (a) $(a, b) = 2^3 \cdot 5^3$

   (b) $[a, b] = 2^5 \cdot 3^7 \cdot 5^8 \cdot 13^2$

(c)

$$a^3b = (2^3 \cdot 5^8 \cdot 13^2)^3(2^5 \cdot 3^7 \cdot 5^3)$$
$$= 2^9 \cdot 2^5 \cdot (5^{24+3} \cdot 3^7 \cdot 13^6) = 2^{14}(5^{27}3^7 13^6).$$

Thus, $e = 14$.

(d)

$$a + b = 2^3 5^8 13^2 + 2^5 3^7 5^3$$
$$= 5^3(2^3 5^5 13^2 + 2^5 3^7).$$

If $5 \mid (2^3 5^5 13^2 + 2^5 3^7)$ then 5 divides $(2^3 5^5 13^2 + 2^5 3^7) - 2^3 5^5 13^2 = 2^5 3^7$. But $5 \nmid 2^5 3^7$ so $5 \nmid (2^3 5^5 13^2 + 2^5 3^7)$ and hence the highest power of 5 that divides $a + b$ is $f = 3$.

◀

4. [**18 Points**] Let $p$ and $q$ denote distinct primes and recall that

$\tau(n)$ = the number of positive divisors of $n$ $\sigma(n)$ = the sum of the positive divisors of $n$.

(a) Find the prime factorization of 350. $350 = 2 \cdot 5^2 \cdot 7$

(b) $\tau(350) = (1 + 1)(1 + 1)(1 + 2) = 12$

(c) $\sigma(350) = (1 + 2)(1 + 5 + 5^2)(1 + 7) = 3 \cdot 8 \cdot 31 = 744$

(d) $\tau(p^4q^2) = (1 + 4)(1 + 2) = 15$

(e) Give the positive divisors of $p^4q^2$ (table form is fine).

| | $1$ | $p$ | $p^2$ | $p^3$ | $p^4$ |
|---|---|---|---|---|---|
| $1$ | $1$ | $p$ | $p^2$ | $p^3$ | $p^4$ |
| $q$ | $q$ | $qp$ | $qp^2$ | $qp^3$ | $qp^4$ |
| $q^2$ | $q^2$ | $q^2p$ | $q^2p^2$ | $q^2p^3$ | $q^2p^4$ |

(f) What are the prime factorizations possible for $n$ if $\tau(n) = 10$. $10 = 2 \cdot 5 = (1 + 1)(1 + 4)$ and $10 = (1 + 9)$. Thus the possible prime factorizations of $n$ are $n = pq^4$ and $n = p^9$ where $p$ and $q$ are distinct primes.

5. [**12 Points**] Use the definition of divisibility to prove that if $a \mid b$ and $a \mid (c - 3b)$ then $a \mid c$.

▶ **Solution.** If $a \mid b$ and $a \mid (c - 3b)$, then from the definition of divisibility, $b = ar$ and $c - 3b = as$ for some integers $r$ and $s$. Then $c = 3b + as = 3ar + as = a(3r + s) = at$ for the integer $t = 3r + s$. Thus, $a \mid c$ by the definition of divides. ◀

6. [**12 Points**] Prove that $2^{n+2} \mid (2n + 3)!$ for all positive integers $n$.

(Recall that $m! = \prod_{j=1}^{m} j = m(m - 1)(m - 2) \cdots 2 \cdot 1$.)

► **Solution.** The proof is by induction on $n \geq 1$. Let $S$ be the set of all integers for which $2^{n+2} \mid (2n+3)!$

***Base Step.*** $1 \in S$ since $2^{1+2} = 2^3 = 8$, $(2 \cdot 1 + 3)! = 5! = 120$, and $8 \mid 120$.

***Induction Step.*** Assume that $k \in S$. That is assume that $2^{k+2} \mid (2k+3)!$. This means that we are assuming that $(2k+3)! = 2^{k+2}m$ for some integer $m$. Then

$$\begin{aligned}
(2(k+1)+3)! &= (2k+5)! \\
&= (2k+5)(2k+4)(2k+3)! \qquad \text{(by the definition of factorial)} \\
&= (2k+5)(2k+4) \cdot 2^{k+2}m \qquad \text{(by the induction hypothesis)} \\
&= 2 \cdot 2^{k+2} \cdot (2k+5)(k+2)m \\
&= 2^{(k+1)+2} \cdot m'
\end{aligned}$$

where $m' = (2k+5)(k+2)m$ is an integer. Thus, $2^{(k+1)+2} \mid (2(k+1)+3)!$ if $2^{k+2} \mid (2k+3)!$. That is, if $k \in S$, then $k+1 \in S$. By the induction principle, $n \in S$ for all $n \geq 1$, so $2^{n+2} \mid (2n+3)!$ for all $n \geq 1$. ◄

7. [**14 Points**] Circle True (T) or False (F). Reasons are not required for this problem.

| | | |
|---|---|---|
| T | F | $a \mid 0$ for all non-zero integers $a$. $0 = a \cdot 0$ for all $a$. |
| T | F | If $a \nmid b$ and $a \nmid c$ then $a \nmid bc$. $6 \nmid 2$ and $6 \nmid 3$ but $6 \mid 2 \cdot 3$. |
| T | F | If $a \mid d$ and $b \mid d$ then $(a, b) \mid d$. $(a, b) \mid a$ and $a \mid d$, so $(a, b) \mid d$. |
| T | F | If $d \mid a$ and $d \mid b$ then $d \mid (a, b)$. Theorem 2.5 |
| T | F | Any integer can be written as a linear combination of 12 and 15. $1 \neq 12x + 3y$ $((12, 15) \neq 1)$. |
| T | F | $496 = 2^4 \cdot 31$ is a perfect number. Theorem 3.11. |
| T | F | If $\tau(n) = 6$ then $n = pq^2$ for distinct primes $p$ and $q$. Could also be of form $p^5$. |