Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**24 Points**] Let $A = \begin{bmatrix} 1 & 2 & -1 & 0 & -3 \\ 2 & 4 & -2 & 1 & -4 \\ -1 & -2 & 1 & 0 & 3 \end{bmatrix}$ and let $R = \begin{bmatrix} 1 & 2 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then R is the reduced row-echelon matrix associated to A.

   (a) Solve the linear system for which A is the augmented matrix.

   (b) Solve the homogeneous linear system for which A is the coefficient matrix.

   (c) What is the rank of A?

   (d) What is the nullity of A?

   (e) Compute a basis for the nullspace $\mathrm{Null}(A)$.

   (f) Compute a basis for the row space $\mathrm{Row}(A)$.

   (g) Compute a basis for the column space $\mathrm{Col}(A)$.

2. [**12 Points**] Find conditions that b_1, b_2, and b_3 must satisfy for the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= b_1 \\ 2x_1 + 3x_2 + 3x_3 &= b_2 \\ 5x_1 + 9x_2 - 6x_3 &= b_3 \end{aligned}$$

   to be consistent.

3. [**12 Points**] Consider the following three vectors in $\mathbb{R}^3$: $\mathbf{u} = (1, 3, 1)$, $\mathbf{v} = (4, 2, -1)$, and $\mathbf{w} = (-3, 1, 2)$.

   (a) Determine whether $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}$ and $\mathbf{w}$.

   (b) Are the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ linearly dependent, or linearly independent?

4. [**8 Points**] Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that $T(\mathbf{e}_1) = (2, -1, 1)$, $T(\mathbf{e}_2) = (1, 3, 2)$, and $T(\mathbf{e}_3) = (1, 4, 2)$, where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ are the standard basis vectors of $\mathbb{R}^3$. Compute $T((4, -2, 5))$.

5. [**12 Points**] Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

6. [**8 Points**] Let A be the 3×3 matrix $A = \begin{bmatrix} 1 & 8 & 3 \\ x & y & z \\ -3 & 7 & 2 \end{bmatrix}$. Assuming that $\det(A) = 5$, compute the determinant of the following matrix

$$B = \begin{bmatrix} x & y & z \\ 3 & 24 & 9 \\ -3 + 4x & 7 + 4y & 2 + 4z \end{bmatrix}.$$

7. [**16 Points**] If $A = \begin{bmatrix} 4 & 1 \\ -3 & 8 \end{bmatrix}$ find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

8. [**12 Points**] Let A be the matrix $A = \begin{bmatrix} -2 & 3 & -1 \\ 0 & 2 & 3 \\ 0 & -1 & 6 \end{bmatrix}$.

   (a) Compute the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$.

   (b) Determine the eigenvalues for A.

   (c) From Part (b) can you determine if A is diagonalizable? Explain, by referring to an appropriate theorem.

9. [**10 Points**] Let $T : V \to W$ be a linear transformation from a vector space V of dimension 6 to a vector space W of dimension 4. Let r be the dimension of the range of T and let s be the dimension of the kernel of T.

   (a) What is the formula relating r and s?

   (b) Suppose there are linearly independent vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in W with $\mathbf{w}_1 = T(\mathbf{v}_1)$ and $\mathbf{w}_2 = T(\mathbf{v}_2)$ for some $\mathbf{v}_1$ and $\mathbf{v}_2$ in V. What are the possible values for r in this case? For each possible value of r, what is the corresponding value for s?

10. [**12 Points**]

   (a) Verify that $\mathcal{B} = \{1,\, 1 - x,\, 1 + x + x^2\}$ is a basis of the vector space P_2 of polynomials of degree ≤ 2.

   (b) Let $p(x) = 2 + x + 3x^2$. Compute the coordinate matrix $[p(x)]_{\mathcal{B}}$.

11. [**12 Points**] Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix}$. Determine if $\mathbf{b}$ is in the column space of A. If it is, express $\mathbf{b}$ as a linear combination of the column vectors of A.

12. [**12 Points**] For each of the following sets of vectors, determine if it is a subspace.

   (a) The set of vectors $(x_1,\, x_2,\, x_3) \in \mathbb{R}^3$ with the property $x_1 + x_2 = 0$.

   (b) The set vectors $(x_1,\, x_2,\, x_3) \in \mathbb{R}^3$ with the property $x_1^2 = x_2^2$.

   (c) The set of all vectors in $\mathbb{R}^3$ which have the form $(a + b,\, a - b,\, c)$ where a, b, and c are arbitrary real numbers.

   (d) The set of all vectors $(x_1,\, x_2,\, x_3) \in \mathbb{R}^3$ with the property $x_1 \geq 0$ and $x_2 \geq 0$.