1. Use Lagrange Multipliers to find the global *maximum and minimum* values of $f(x, y) = x^2 + 2y^2 - 4y$ subject to the constraint $x^2 + y^2 = 9$.

   ▶ **Solution.** Let $g(x, y) = x^2 + y^2$. Then solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$, $g(x, y) = k$ to find the critical points:

$$\begin{aligned} 2x &= \lambda 2x \\ 4y - 4 &= \lambda 2y \\ x^2 + y^2 &= 9. \end{aligned}$$

   The first equation gives $x(\lambda - 1) = 0$ so $x = 0$ or $\lambda = 1$. If $x = 0$, then the third equation gives $y^2 = 9$ so $y = \pm 3$. If $\lambda = 1$ then the second equation gives $4y - 4 = 2y$ so $y = 2$. The third equation then gives $x^2 = 2^2 = 9$ so $x = \pm\sqrt{5}$. Thus, there are four critical points: $(0, \pm 3)$ and $(\pm\sqrt{5}, 2)$. Evaluating $f(x, y)$ at these points gives: $f(0, 3) = 6$, $f(0, -3) = 30$, $f(\sqrt{5}, 2) = 5 = \mathbb{F}(-\sqrt{5}, 2)$. Therefore, the global maximum is 30, which occurs at $(0, -3)$, and the global minimum is 5, which occurs at the two points $(\pm\sqrt{5}, 2)$. ◀

2. Compute $\displaystyle\int_0^2 \int_{y^2}^{2y} (4x - y)\, dx\, dy$.

   ▶ **Solution.**

$$\begin{aligned} \int_0^2 \int_{y^2}^{2y} (4x - y)\, dx\, dy &= \int_0^2 (2x^2 - xy)\Big|_{x=y^2}^{x=2y}\, dy \\ &= \int_0^2 (6y^2 - 2y^4 + y^3)\, dy = \left(2y^3 - \frac{2y^5}{5} + \frac{y^4}{4}\right)\Big|_0^2 \\ &= 16 - \frac{64}{5} + 4 = \frac{36}{5}. \end{aligned}$$

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3. Compute $\displaystyle\int_0^1 \int_0^x \int_0^{xy} xyz\, dz\, dy\, dx$.

   ▶ **Solution.**

$$\begin{aligned} \int_0^1 \int_0^x \int_0^{xy} xyz\, dz\, dy\, dx &= \int_0^1 \int_0^x \frac{xyz^2}{2}\Big|_{z=0}^{z=xy}\, dy\, dx \\ &= \int_0^1 \int_0^x \frac{x^3y^3}{2}\, dy\, dx = \int_0^1 \frac{x^3y^4}{8}\Big|_{y=0}^{y=x}\, dx \\ &= \int_0^1 \frac{x^7}{7}\, dx = \frac{x^8}{64}\Big|_0^1 = \frac{1}{64}. \end{aligned}$$

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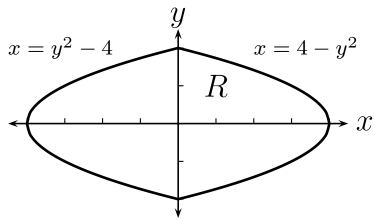
4. Let R be the region in the plane bounded by the graphs of $y^2 = 4 + x$ and $y^2 = 4 - x$.

   (a) Sketch R.

   ► **Solution.**



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   (b) If $f(x, y)$ is an arbitrary continuous function defined on R, express $\iint_R f(x, y)\, dA$ as an iterated double integral.

   ► **Solution.**

$$\iint_R f(x, y)\, dA = \int_{-2}^{2} \int_{y^2-4}^{4-y^2} f(x, y)\, dx\, dy$$

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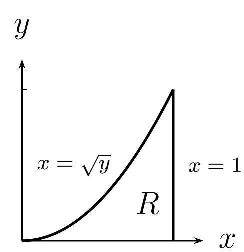
5. Compute the following integral:

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(x^3)\, dx\, dy.$$

(*Hint:* First draw the domain of integration. Then reverse the order of integration.)

► **Solution.** First draw the domain of integration R:



Then write the curve $x = \sqrt{y}$ as $y = x^2$ and change the order of integration

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(x^3)\, dx\, dy = \int_0^1 \int_0^{x^2} \sin(x^3)\, dy\, dx = \int_0^1 y \sin(x^3)\Big|_0^{x^2} dx$$

$$= \int_0^1 x^2 \sin(x^3)\, dx = -\frac{1}{3}\cos(x^3)\Big|_0^1 = \frac{1}{3}(1 - \cos 1).$$

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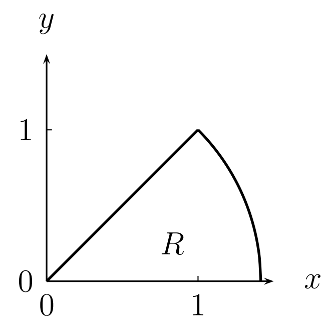
6. Compute the following integral:

$$\int_0^1 \int_y^{\sqrt{2-y^2}} \sqrt{x^2+y^2}\, dx\, dy.$$

(*Hint:* First draw the domain of integration. Then use polar coordinates.)

▶ **Solution.** First draw the domain of integration R:



Then R can be expressed in polar coordinates as $0 \le \theta \le \pi/4$, $0 \le r \le \sqrt{2}$. Then $dA = r dr\, d\theta$ and

$$\begin{aligned}\int_0^1 \int_y^{\sqrt{2-y^2}} \sqrt{x^2+y^2}\, dx\, dy &= \iint_R \sqrt{x^2+y^2}\, dA \\ &= \int_0^{\pi/4} \int_0^{\sqrt{2}} r\, r dr\, d\theta = \int_0^{\pi/4} \int_0^{\sqrt{2}} r^2\, dr\, d\theta \\ &= \int_0^{\pi/4} \left.\frac{r^3}{3}\right|_{r=0}^{r=\sqrt{2}} d\theta = \int_0^{\pi/4} \frac{2\sqrt{2}}{3} d\theta \\ &= \frac{2\sqrt{2}\pi}{12} = \frac{\sqrt{2}\pi}{6}.\end{aligned}$$

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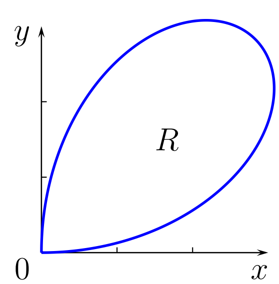
7. Compute the area of one leaf of the four leaved rose $r = a\sin(2\theta)$.

▶ **Solution.** First draw a picture of one leaf:



Then the single leaf can be expressed in polar coordinates as $0 \le \theta \le \frac{\pi}{2}$, $0 \le r \le a\sin 2\theta$, and the area is given by

$$\begin{aligned} Area &= \iint_R dA = \int_0^{\pi/2}\int_0^{a\sin 2\theta} r\,dr\,d\theta = \int_0^{\pi/4} \frac{r^2}{2}\bigg|_0^{a\sin 2\theta} d\theta \\ &= \frac{a^2}{2}\int_0^{\pi/2} \sin^2 2\theta\,d\theta = \frac{a^2}{2}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2}\,d\theta \\ &= \frac{a^2}{4}\left(\theta - \frac{\sin 4\theta}{4}\right)\bigg|_0^{\pi/2} \\ &= \frac{a^2\pi}{8}. \end{aligned}$$

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8. Compute the volume of the region in the first octant that is bounded by the coordinate planes and the plane $x + y + z = 3$.

   ▶ **Solution.** The region is above the triangular region in the xy-plane bounded by the axes and the line $x + y = 3$, and it is below the plane $z = 3 - x - y$. Thus, the volume is given by

$$\begin{aligned} \int_0^3\int_0^{3-y}(3-x-y)\,dx\,dy &= \int_0^3 \left(3x - \frac{x^2}{2} - xy\right)\bigg|_{x=0}^{x=3-y} dy \\ &= \int_0^3 \left(9 - 3y - \frac{9-6y+y^2}{2} - (3y-y^2)\right) dy \\ &= \int_0^3 \left(\frac{9}{2} + \frac{y^2}{2} - 3y\right) dy = \left(\frac{9}{2}y + \frac{y^3}{6} - \frac{3y^2}{2}\right)\bigg|_0^3 \\ &= \frac{27}{2} + \frac{27}{6} - \frac{27}{2} = \frac{27}{6} = \frac{9}{2}. \end{aligned}$$

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9. Compute the volume of the finite region Q bounded by the graphs of $z = 9 - x^2 - y^2$, $x^2 + y^2 = 4$, and $z = 0$. Use cylindrical coordinates.

   ▶ **Solution.** The volume of Q is

$$\begin{aligned} \iiint_Q dV &= \int_0^{2\pi}\int_0^2\int_0^{9-r^2} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 z\big|_{z=0}^{z=9-r^2} r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (9r - r^3)\,dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{9r^2}{2} - \frac{r^4}{4}\right)\bigg|_0^2 d\theta = \int_0^{2\pi} 14\,d\theta \\ &= 28\pi. \end{aligned}$$

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10. Let Q be the region bounded below by the cone $z^2 = x^2 + y^2$ and above by the sphere of radius $\sqrt{2}$ and center at the origin. Compute the volume of Q using spherical coordinates.

► **Solution.** The cone and the sphere intersect when $x^2 + y^2 = z^2 = 2 - x^2 - x^2$ so $x^2 + y^2 = 1$. In Q the z-coordinate is positive. The cone is defined in spherical coordinates by $\phi = \pi/4$ and Q is symmetric around the z-axis. Thus, Q is defined in spherical coordinates by $0 \le \rho \le \sqrt{2}$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/4$. Hence, the volume of Q is given by the integral

$$\begin{aligned}
\iiint_Q dV &= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \frac{\rho^3}{3} \sin\phi \bigg|_0^{\sqrt{2}} d\theta \, d\phi = \frac{2\sqrt{2}}{3} \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \sin\phi \, d\theta \, d\phi \\
&= (2\pi)\left(\frac{2\sqrt{2}}{3}\right) \int_0^{\frac{\pi}{4}} \sin\phi \, d\phi = (2\pi)\left(\frac{2\sqrt{2}}{3}\right) (-\cos\phi)\big|_0^{\pi/4} \\
&= \frac{4\pi}{3}(\sqrt{2} - 1).
\end{aligned}$$

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