A Course on the Yosida Theorem

Classical & Pointfree Versions & Applications

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Summer 2020

Lecture 9

More Computations:
Proof of Localic Yosida Concluded

Tuesday, August 4, 2020

Review: Where we stand, now:

Let A be an Archimedean ℓ-group, and suppose $e \in A^+$.

We have defined:

(1) the "general Yosida locale" $\mathcal{Y}A$ and the "topped Yosida locale" $\mathcal{Y}(A, e)$.

(2) for each $a \in A$, the frame morphism $\Phi(a, e) : \mathcal{R} \to \mathcal{Y}(A, e)$, and hence the map: $\Phi(\_, e) : A \to \mathcal{R}\,\mathcal{Y}(A, e)$.

In the last lecture, we indicated some of the argument needed to show that $\mathcal{R}\,\mathcal{Y}(A, e)$ is an ℓ-group. We leave the full verification to the audience. (See "Madden-1992-frames.pdf" for hints.)

Comment on notation. We may write $\mathcal{Y}_e A$ instead of $\mathcal{Y}(A, e)$ and $\Phi_e(a)$ instead of $\Phi(a, e)$. The map in (2), above, may be written $\Phi_e : A \to \mathcal{R}\,\mathcal{Y}_e A$.

The next step (cf. Lecture 7, slide 8, item 4) in our proof-sketch of the localic Yosida Theorem (constructive version) is to show that $\Phi_e(a + b) = \Phi_e(a) + \Phi_e(b)$ for all $a, b \in A$. One must also verify that Φ_e preserves $\vee$, but this is similar to (and easier than) showing that Φ_e preserves $+$.

The classical Yosida Representation starts with algebraic data (maximal ideals) and then introduces a topology. The localic version inverts this: it starts with a localic construction, then shows that the algebraic structures carry through.

What we need to prove

Pointwise addition of functions is described by the following diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{\Delta} & X \times X & \xrightarrow{f \times g} & \mathbb{R} \times \mathbb{R} & \xrightarrow{+} & \mathbb{R}, \\ x & \longmapsto & (x, x) & \longmapsto & (f(x), g(x)) & \longmapsto & f(x) + g(x). \end{array}$$

The corresponding operation in frames is described thus (where $\phi, \gamma : \mathcal{R} \to \mathcal{O}$):

$$\begin{array}{ccccccc} \mathcal{O} & \xleftarrow{\nabla} & \mathcal{O} \otimes \mathcal{O} & \xleftarrow{\phi \otimes \gamma} & \mathcal{R} \otimes \mathcal{R} & \xleftarrow{\alpha} & \mathcal{R}, \\ \bigvee\{\, \phi(s,t) \wedge \gamma(u,v) \mid \cdots \,\} & \longleftarrow\!\!\shortmid & \bigvee\{\, \phi(s,t) \otimes \gamma(u,v) \mid \cdots \,\} & \longleftarrow\!\!\shortmid & \bigvee\{\, (s,t) \otimes (u,v) \mid \cdots \,\} & \longleftarrow\!\!\shortmid & (p,q), \end{array}$$

where $\cdots$ stands for $p \leqslant s + u \,\&\, t + v \leqslant q$.

To show that $\Phi_e(a + b) = \Phi_e(a) + \Phi_e(b)$, we must show that:

$$\Phi_e(a + b)(p, q) = \bigvee\{\, \Phi_e(a)(s, t) \wedge \Phi_e(b)(u, v) \mid p \leqslant s + u \,\&\, t + v \leqslant q \,\}.$$

The LHS is:

$$y_e\left((a + b - pe)^+ \wedge (qe - (a + b))^+\right).$$

The RHS is:

$$\bigvee_{\text{in } \mathcal{Y}_e A} \left\{ y_e\left((a - se)^+ \wedge (te - a)^+\right) \wedge y_e\left((b - ue)^+ \wedge (ve - b)^+\right) \,\middle|\, p \leqslant s + u \,\&\, t + v \leqslant q \right\}.$$

Note: $(\bigvee A) \wedge (\bigvee B) = \bigvee\left\{ (\bigvee A) \wedge b \,\middle|\, b \in B \right\} = \bigvee\{\, a \wedge b \mid a \in A, b \in B \}$.

What we need to prove (continued)

It suffices to show:

$$
\begin{aligned}
(i) \quad & y_e\left((a+b-pe)^+\right) &=& \bigvee \{\, y_e\left((a-se)^+ \wedge (b-ue)^+\right) \mid p \leqslant s+u \,\},\\
(ii) \quad & y_e\left((qe-(a+b))^+\right) &=& \bigvee \{\, y_e\left((te-a)^+ \wedge (ve-b)^+\right) \mid t+v \leqslant q \,\}.
\end{aligned}
$$

The $\geqslant$ inequalities follow from:

Lemma. Any ℓ-group satisfies the identity: $(x+y)^+ \geqslant x^+ \wedge y^+$.

Proof. $(x+y)^+ - (1/2)(x+y) = (1/2)|x+y|$.
$\left[x^+ \wedge y^+\right] - (1/2)(x+y) = (1/2)\left[(x-y) \wedge (y-x)\right] = -(1/2)|x-y|$. $\square$

We include the proof because it illustrates a strategy that we use again (with g and g_n, next slide).

To show the $\leqslant$ inequalities, it suffices to show:

(*i*) $[(1/2)(a+b-pe)^+ \wedge e] \in RU\{\, (a-se)^+ \wedge (b-ue)^+ \wedge e \mid s+u=p \,\}$, and

(*ii*) $[(1/2)(qe-(a+b))^+ \wedge e] \in RU\{\, (te-a)^+ \wedge (ve-b)^+ \wedge e \mid t+v \leqslant q \,\}$.

Here, $RU\,X$ denotes the set of relative-uniform limits of sequences of elements of X. Note that including the "$\wedge\, e$" is permissible, because $y_e(e) = \top_{\mathcal{Y}_e A}$, and $y_e(a \wedge b) = y_e(a) \wedge y_e(b)$. The factor $(1/2)$ is harmless, because $y_e((1/2)\, a) = y_e(a)$.

Proof of (ii)

Suppose A is a divisible abelian ℓ-group, $a, b \in A$, $e \in A^+$, $q \in \mathbb{Q}$, and $n \in \mathbb{N}$. Let

$$g := \tfrac{1}{2}\left(q\,e - (a+b)\right), \text{ and } \qquad g_n := \bigvee_{i=-n^2}^{n^2} \left[\left(\left(\tfrac{1}{2}q + \tfrac{i}{n}\right) e - a\right) \wedge \left(\left(\tfrac{1}{2}q - \tfrac{i}{n}\right) e - b\right)\right].$$

$$g - g_n = \tfrac{1}{2}\left(q\,e - (a+b)\right) + \bigwedge_{i=-n^2}^{n^2} \left[\left(a - \left(\tfrac{1}{2}q + \tfrac{i}{n}\right) e\right) \vee \left(b - \left(\tfrac{1}{2}q - \tfrac{i}{n}\right) e\right)\right]$$

$$= \bigwedge_{i=-n^2}^{n^2} \left[\left(\tfrac{1}{2}(a-b) - \tfrac{i}{n}e\right) \vee \left(\tfrac{1}{2}(b-a) + \tfrac{i}{n}e\right)\right] = \bigwedge_{i=-n^2}^{n^2} \left|\tfrac{1}{2}(a-b) - \tfrac{i}{n}e\right|. \qquad \textit{Mensch!}$$

Lemma. If $f, w \in A$ and $0 \leqslant w$, then $\bigwedge_{i=-m}^{m} |f - i\,w| \leqslant (|f| - m\,w) \vee w$. ("Madden-1992-frames.pdf", 4.4.)

$$0 \leqslant g - g_n \leqslant \left(\tfrac{1}{2}|a-b| - n\,e\right) \vee \tfrac{1}{n}e$$
$$\leqslant \left(\tfrac{1}{2}|a-b| - n\,e\right)^+ \vee \tfrac{1}{n}e$$
$$(g - g_n)^+ \wedge e \leqslant \left(\left(\tfrac{1}{2}|a-b| - n\,e\right)^+ \wedge e\right) \vee \tfrac{1}{n}e$$

Lemma. If $x, w \in A^+$, $(x - n\,w)^+ \wedge w \leqslant \tfrac{1}{n}x$. ("Madden-1992-frames.pdf", 4.3.)

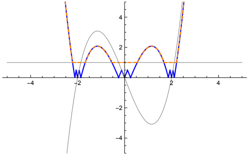
$$(g^+ \wedge e) - (g_n^+ \wedge e) \leqslant (g - g_n)^+ \wedge e \leqslant \tfrac{1}{n}\left(\tfrac{1}{2}|a-b| \vee e\right)$$

This concludes the proof of (ii).

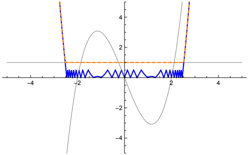
Remarks on a key lemma.

Lemma. If $f, w \in A$ and $0 \leqslant w$, then $\bigwedge_{i=-m}^{m} |f - i\,w| \leqslant (|f| - m\,w) \vee w$.

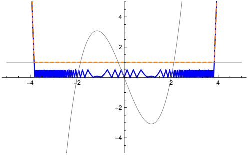
Here are graphs of the functions in the lemma when $w = 1$, $f = x^3 - 4x$ (in gray) and m takes the values 1, 5 and 40. The LHS is depicted in blue, and the RHS in orange.



$m = 1$



$m = 5$



$m = 40$

Summary

We have now completed the proof sketch of the following theorem:

Localic Yosida. Suppose A is an archimedean ℓ-group, and $e \in A^+$. Let $y_e : A^+ \to \mathcal{Y}_e A$ be the frame freely generated by A^+ modulo relations (I_1)-(I_4), (Y) and $y_e(e) = \top$. Let $\Phi_e : A \to \mathcal{R}\,\mathcal{Y}_e A$ be defined by

$$\Phi_e(a)(p, q) = y_e\left((a - pe)^+ \wedge (qe - a)^+\right), \quad p, q \in \mathbb{Q}.$$

Then $\mathcal{Y}_e A$ regular Lindelöf and Φ_e is an ℓ-homomorphism.

What is $\ker \Phi_e$? (Cf. Lecture 3, slides 5 and 8)

Proposition. Suppose A is an archimedean ℓ-group and $a, e \in A^+$. The following are equivalent:

(i) $a \in \ker \Phi_e$; (ii) $y_e(a) = \bot$; (iii) $a \wedge e = 0$.

Remark. Suppose $\phi : \mathcal{R} \to \mathcal{O}$. Then

$$\phi = 0 \;\Leftrightarrow\; \text{for all } p, q \in \mathbb{Q},\ \phi(p, q) = \begin{cases} \top, & \text{if } p < 0 < q; \\ \bot, & \text{otherwise} \end{cases}.$$

Proof. $(i \Rightarrow ii)$ $\Phi_e(a) = 0$ implies $y_e\left((a - pe)^+\right) = \begin{cases} \top, & \text{if } p < 0; \\ \bot, & \text{otherwise} \end{cases}$.

Thus $y_e(a) = y_e\left((a - 0e)^+\right) = \bot$.

$(ii \Rightarrow iii)$ $y_e(a) = y(e) \wedge y(a) = y(e \wedge a)$. Since A is archimedean, $0 \leqslant b$ & $y(b) = 0$ iff $b = 0$. (Caution/Question. "Archimedean" has more than one constructive interpretation. What are we using, here?)

$(iii \Rightarrow i)$ Suppose $a \wedge e = 0$. Then, $y_e(a) = y_e(a \wedge e) = \bot$. Also,
$(a - pe)^+ = \begin{cases} a \vee -pe, & \text{if } p < 0; \\ a, & \text{if } 0 \leqslant p \end{cases}$, and $(qe - a)^+ = \begin{cases} qe, & \text{if } q > 0; \\ 0, & \text{if } q \leqslant 0 \end{cases}$. Thus
$y_e\left((a - pe)^+\right) = \begin{cases} \top, & \text{if } p < 0; \\ \bot, & \text{if } 0 \leqslant p \end{cases}$, and $y_e\left((qe - a)^+\right) = \begin{cases} \top, & \text{if } q > 0; \\ \bot, & \text{if } q \leqslant 0 \end{cases}$. $\square$

Definition. For $a \in A$, $a^{\perp} := \{\, b \in A \mid 0 = |a| \wedge |b| \,\}$. We say a is a *weak unit* if $0 \leqslant a$ and $a^{\perp} = \{0\}$.

Theorem. Suppose A is archimedean, $a, b \in A$ and $0 \leqslant a \leqslant b$. Then $a^{\perp}$ is the intersection of the values M of a such that $b \in M^{*} (:= \langle M, a \rangle)$.

Corollary. If A is archimedean, then $\boxed{a^{\perp} \text{ is the intersection of the values of } a}$, and $\mathrm{fin}_u\, b$ is dense in $Y(A, a)$ for all $b \in A$. ↑this yields classical version of proposition on previous slide.

Proof. If $a \notin M$ then $a^{\perp} \subseteq M$, so $a^{\perp} \subseteq \bigcap Val(A, a)$. To prove the opposite inclusion, suppose $0 \leqslant x \notin a^{\perp}$. Then, $0 < x \wedge a$. Let $d := x \wedge a$. Note that $d \leqslant b$. Since A is archimedean and $0 < d$, we may—and do—pick $n \in \mathbb{N}$ such that $n\, d \nleqslant b$. Let

$$h := b - (n\, d \wedge b), \text{ and } g := n\, d - (n\, d \wedge b).$$

Note that $g \wedge h = 0$. Pick P maximal missing g. Since P is prime, $h \in P$. Also, P does not contain b (otherwise, it would contain g, because $0 < n\, d \leqslant n\, b$ and $(n\, d \wedge b) \leqslant b$). Enlarge P to a value M of b. Since $b \notin M$ but $h \in M$, $n\, d \wedge b \notin M$, so $d \notin M$, so neither x nor a is in M. Clearly $a \in \langle M, b \rangle = M^{*}$, so M is a value of a. □

What's next?

1. Suppose $e, f \in A^+$. What is the relationship between Φ_e and Φ_f?
2. Some thoughts about representations of **Arch**.
3. Examples of some representations:
   - Free divisible abelian ℓ-groups
   - Finitely-supported functions on $\mathbb{N}$
   - Conrad-Martinez example
   - Countably-supported functions on an uncountable set
   - Hager's Unusual Epicomplete Archimedean ℓ-group ("Hager-2015-unusual.pdf")