A Course on the Yosida Theorem

Classical & Pointfree Versions & Applications

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Lecture 8
Computations in the Locale of Real Numbers

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Review of conventions on presentations of frames

X	a set
$i : X \to \mathcal{F}X$	the free frame on X, with i the embedding of the generators
R	a relation on $\mathcal{F}X$ (often written as a set of equations)
$\mathcal{F}X/R$	the quotient of $\mathcal{F}X$ by the congruence generated by R
$i_R : X \to \mathcal{F}X/R$	i followed by the quotient map $\mathcal{F}X \to \mathcal{F}X/R$.

Let $\mathcal{G}$ be a frame. We say a set map $\phi : X \to \mathcal{G}$ *is an R-map* if for every equation $E(x) \in R$, the equation $E(\phi(x))$ (obtained by replacement) is true in $\mathcal{G}$.

We have shown that i_R has the following universal mapping property: if $\phi : X \to \mathcal{G}$ is an R-map, then there is a unique morphism $\overline{\phi} : \mathcal{F}X/R \to \mathcal{G}$ such that $\overline{\phi} \circ i_R = \phi$. We call $i_R : X \to \mathcal{F}X/R$ the *the universal R-map*, and we say that $\mathcal{F}X/R$ is the *free frame on X subject to R*.

Using this strategy, we have defined:

- for any **Arch**-object A and $e \in A^+$, the Yosida locale $\mathcal{Y}(A, e)$,
- the frame of reals $\mathcal{R}$,
- for each $a \in A$, a frame morphism $\Phi(a, e) : \mathcal{R} \to \mathcal{Y}(A, e)$.

This gives us a map: $\Phi(\_, e) : A \to \mathcal{R}\,\mathcal{Y}(A, e)$.* Our goal now is to show:

(i) $\mathcal{R}\,\mathcal{Y}(A, e)$ is an ℓ-group,
(ii) $\Phi(\_, e)$ is an ℓ-group homomorphism.

* For any frame $\mathcal{G}$, $\mathcal{R}\,\mathcal{G}$ denotes the set of frame morphisms from $\mathcal{R}$ to $\mathcal{G}$.

Localic Algebras

Localic algebras. For any frame $\mathcal{A}$, let $L(\mathcal{A})$ denote the corresponding locale. An n-ary localic operation on $L(\mathcal{A})$ is a function from the n-fold localic product of $L(\mathcal{A})$ with itself to $L(\mathcal{A})$:

$$f : L(\mathcal{A}) \times_{loc} \cdots \times_{loc} L(\mathcal{A}) \to L(\mathcal{A}),$$

i.e., a frame map:

$$\mathcal{A} \to \mathcal{A} \otimes \cdots \otimes \mathcal{A}.$$

Review of frame coproducts. Let $\mathcal{A}$ and $\mathcal{B}$ be frames. The *coproduct* $\mathcal{A} \otimes \mathcal{B}$ is the free frame on (the set) $\mathcal{A} \times \mathcal{B}$ subject to the following relations, with $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $X \subseteq \mathcal{A}$ and $Y \subseteq \mathcal{B}$:

(C_1) $\quad i(\top, \top) = \top$; $i(a, \bot) = i(\bot, b) = \bot$;
(C_2) $\quad i(a, b) \wedge i(a', b') = i(a \wedge a', b \wedge b')$;
(C_3) $\quad \bigvee\{\, i(x, b) \mid x \in X \,\} = i(\bigvee X, b)$; $\bigvee\{\, i(a, y) \mid y \in Y \,\} = i(a, \bigvee Y)$.

Each element of $\mathcal{A} \otimes \mathcal{B}$ may be expressed in the form $\bigvee\{\, a_i \otimes b_i \mid i \in I \,\}$.

A *localic algebraic structure* is a locale equipped with a set of operations. Further facts about localic algebraic structures—e.g., Ω-algebra-objects, algebra morphisms, equational laws—will be introduced as needed.

Defining addition in $L(\mathcal{R})$

We define $\alpha_0 : \mathbb{Q}^2 \to \mathcal{R} \otimes \mathcal{R}$ by:

$$\alpha_0(p, q) = \bigvee A(p, q)$$

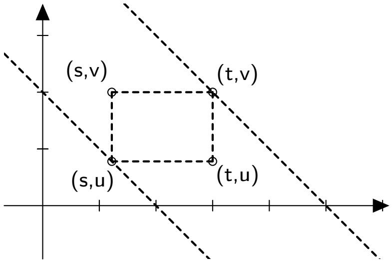
where

$$A(p, q) := \{ (s, t) \otimes (u, v) \mid p \leqslant s + u \;\&\; t + v \leqslant q \}$$

We will show that α_0 satisfies the Joyal relations (which we repeat on the next slide for reference).

We are going to use a geometric argument, which depends on the observation that there is a bijection between the non-trivial elements of $A(p, q)$ and the open rational rectangles in the strip $p < x + y < q$.

Caution. Let $\mathbf{R}$ be a collection of rational rectangles. If $\mathbf{s}$ is rational rectangle such that $\mathbf{s} \subseteq \bigcup \mathbf{R}$ (sets-theoretic union), it need NOT be the case that $\mathbf{s} \leqslant \bigvee \mathbf{R}$ in $\mathcal{R} \otimes \mathcal{R}$. The defining relations for $\mathcal{R} \otimes \mathcal{R}$ allow us to join rectangles that have a common edge, but not arbitrary collections.



Verifying the Joyal relations for α_0: (R_0) and (R_1)

The Joyal relations for a function $f : \mathbb{Q}^2 \to \mathcal{A}$ are as follows. ($\mathcal{R}$ is the free frame on $\mathbb{Q}^2$ subject to these.)

(R_0)	If $q \leqslant p$, then $f(p,q) = \perp$;	(Redundant—follows from (R_3))
(R_1)	$f(p_1,q_1) \wedge f(p_2,q_2) = f(p_1 \vee p_2, q_1 \wedge q_2)$;	
(R_2)	If $p_1 \leqslant p_2 < q_1 \leqslant q_2$, then $f(p_1,q_1) \vee f(p_2,q_2) = f(p_1,q_2)$;	
(R_3)	$\bigvee\{ f(x,y) \mid x,y \in \mathbb{Q} \;\&\; p < x < y < q \} = f(p,q)$;	
(R_4)	$\bigvee\{ f(x,y) \mid x,y \in \mathbb{Q} \} = \top$.	

We verify $(R_0$-$R_4)$ for $f = \alpha_0(p,q) := \bigvee\{ (s,t) \otimes (u,v) \mid p \leqslant s+u \;\&\; t+v \leqslant q \}$:

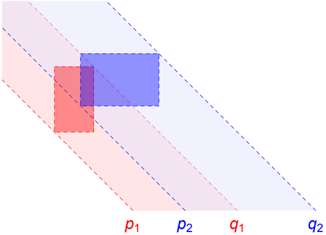
(R_0): If $t+v \leqslant q \leqslant p \leqslant s+u$, then $s \geqslant t$ or $v \geqslant u$, so $(s,t) \otimes (u,v) = \perp$ in $\mathcal{R} \otimes \mathcal{R}$.

(R_1): We express LHS using distributivity and compare to RHS:

$$LHS = \bigvee\{ (s_1 \vee s_2, t_1 \wedge t_2) \otimes (u_1 \vee u_2, v_1 \wedge v_2) \mid p_i \leqslant s_i + u_i \;\&\; t_i + v_i \leqslant q_i \; (i = 1,2) \}$$

$$RHS = \bigvee B; \; B := \{ (s,t) \otimes (u,v) \mid p_1 \vee p_2 \leqslant s+u \;\&\; t+v \leqslant q_1 \wedge q_2 \}$$

The red and blue rectangles are in $A(p_1,q_1)$ and $A(p_2,q_2)$, respectively. Their intersection (purple) is one of the $\vee$-terms in LHS. The $\vee$-terms of RHS are the rectangles between $x+y=p_2$ and $x+y=q_1$, like the purple one. (We have drawn maximal rectangles, but the argument is the same for any.)



Verifying the Joyal relations for α_0: (R_2)

(R_2) Suppose $p_1 \leqslant p_2 < q_1 \leqslant q_2$. Then $\alpha_0(p_1, q_1) \vee \alpha_0(p_2, q_2) = \alpha_0(p_1, q_2)$.

Lemma. Suppose $p, q \in \mathbb{Q}$ and $\epsilon \in \mathbb{Q}_{>0}$. Then, in $\mathcal{R}$,

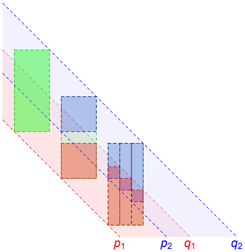
$$(p, q) = \bigvee \{ (x, y) \mid p < x < y < q \;\&\; y - x < \epsilon \}.$$

Proof of (R_2). Put $\epsilon = \frac{1}{2}(q_1 - p_2)$, and put

$$S = \{ (s, t) \oplus (u, v) \mid p_1 \leqslant s + u \;\&\; t + v \leqslant q_2 \;\&\; t - s < \epsilon \;\&\; v - u < \epsilon \}.$$

By the Lemma, $\alpha_0(p_1, q_2) = \bigvee S$. But every rectangle in S either has its upper right vertex on or below $x + y = q_1$ or its lower left vertex on or above $x + y = p_2$. □

Geometric view. We need to cover the green rectangle with rectangles that lie either entirely within the red strip or entirely within the blue strip. The second image shows that maximal rectangles will not do the job. (C_3) says that we must build up covers with sets of rectangles that have a fixed projection onto one of the axes. The third image shows that sufficiently narrow rectangles (with height equal to the green rectangle) can be covered by rectangles with a common projection onto the x-axis, which are entirely within the red strip or entirely within the blue strip. These all have the same height, and they cover the green rectangle. (In the proof, we consider rectangles whose dimensions are less than half the horizontal (or vertical) width of the overlap.)



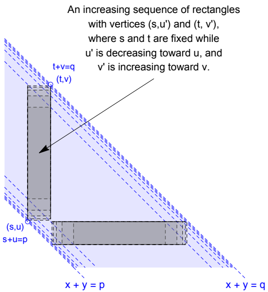
Verifying the Joyal relations for α_0: (R_3)

$$(R_3): \quad \alpha_0(p,q) := \bigvee \left\{ (s,t)\otimes(u,v) \;\middle|\; p \leqslant s+u \;\&\; t+v \leqslant q \right\}$$

$$= \bigvee \left\{ (s',t')\otimes(u',v') \;\middle|\; \exists p', q' \;\; \begin{array}{r} p<p'<q'<q \\ p' \leqslant s'+u' \\ t'+v' \leqslant q' \end{array} \right\}$$

$$= \bigvee \left\{ \alpha_0(p',q') \;\middle|\; p < p' < q' < q \right\}$$

Explanation. The first equality is the definition. The second is due to the fact that any rectangle in the strip between $x+y=p$ and $x+y=q$ $(p<q)$ is covered by an increasing sequence of rectangles with the same projection onto one axis, lying in strips between $x+y=p'$ and $x+y=q'$ $(p<p'<q'<q)$, as shown in the picture. (This follows from (R_3) for $\mathcal{R}$ and (C_3).) The third equality follows from the fact that the set of terms in the supremum on the second line is the union of the sets of terms that appear in the expressions for $\alpha_0(p',q')$, $p<p'<q'<q$.



Verifying the Joyal relations for α_0: (R_4)

Reminder: $\alpha_0(p,q) := \bigvee\{\,(s,t)\otimes(u,v)\mid p\leqslant s+u\ \&\ t+v\leqslant q\,\}$

$$
\begin{aligned}
(R_4):\quad \bigvee\{\,\alpha_0(p,q)\mid p,q\in\mathbb{Q}\,\} &= \bigvee\{\,(s,t)\otimes(u,v)\mid s,t,u,v\in\mathbb{Q}\,\}\\
&\geqslant \bigvee\left\{\bigvee\{\,(s,t)\otimes(u,v)\mid s,t\in\mathbb{Q}\,\}\ \middle|\ u,v\in\mathbb{Q}\right\}\\
&= \bigvee\{\,\top_{\mathcal{R}}\otimes(u,v)\mid u,v\in\mathbb{Q}\,\}\\
&= \top_{\mathcal{R}\otimes\mathcal{R}}
\end{aligned}
$$