A Course on the Yosida Theorem

Classical & Pointfree Versions & Applications

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Lecture 7
(a) Lindelöf Locales
(b) The Locale of Real Numbers

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Compact and Lindelöf elements of a frame

An element a of a frame is said to be:

- *compact* if: $a \leqslant \bigvee B \Rightarrow \exists$ finite $B' \subseteq B$ s.t. $a \leqslant \bigvee B'$;
- *Lindelöf* if: $a \leqslant \bigvee B \Rightarrow \exists$ countable $B' \subseteq B$ s.t. $a \leqslant \bigvee B'$.

A frame is said to be compact (Lindelöf) if its top element is such.

If $\mathit{Idl}_\ell\, A$:= the frame of ℓ-ideals of an abelian ℓ-group A, the compact elements of $\mathit{Idl}_\ell\, A$ are the finitely-generated (= principal) ℓ-ideals. $\mathit{Idl}_\ell\, A$ is compact iff A has an element that is contained in no proper ℓ-ideal, i.e., a strong unit.

Let $\mathit{ArchK}\, A$:= the frame of archimedean kernels of A. The Lindelöf elements of $\mathit{ArchK}\, A$ are the countably-generated archimedean kernels. $\mathit{ArchK}\, A$ is Lindelöf if there is countable $B \subseteq A$ that is contained in no proper archimedean kernel.

Example. Let A denote the ℓ-group of sequences with finite support. A has no weak unit, but nonetheless $\mathit{ArchK}\, A$ is Lindelöf.

Example. If $\mathcal{Y}(A, e) = (\mathit{ArchK}\, A)/(ru\langle e\rangle \sim \top)$ is Lindelöf, because $ru\langle e\rangle$ is.

In general, $\mathcal{Y}(A, e)$ need not be compact. The countable relation Y allows that $ru\langle e\rangle$ may be the supremum of a countable family of archimedean kernels strictly smaller than $ru\langle e\rangle$.

A Digression on Coherent and Algebraic Frames

There are several different categories of distributive lattices, depending on what structure is present and preserved by morphisms.

- $\mathbf{D}_0^1$ denotes the category of distributive lattices with bottom and top element and 0-1-$\vee$-$\wedge$-preserving morphisms.
- $\mathbf{D}_0$ denotes the category of distributive lattices with bottom element and 0-$\vee$-$\wedge$-preserving morphisms.

To each of these categories, there is a forgetful functor from **Frm**. Each has a left adjoint (because free frames exist).

The left adjoint of the forgetful functor $\mathbf{Frm} \to \mathbf{D}_0^1$ assigns to a 0-1-lattice D its frame of ideals, denoted by *$Idl\,D$*. The top of $Idl\,D$ is $\downarrow 1_D = D$, and the bottom is $\downarrow 0_D = \{0_D\}$. A frame of the form $Idl\,D$ is said to be *coherent*. The coherent frames are characterized as those whose the compact elements form a generating sub-0-1-lattice. See Johnstone, *Stone Spaces*, for a discussion.

The following remark is relevant the "unknown representation". The left adjoint of the forgetful functor $\mathbf{Frm} \to \mathbf{D}_0$ assigns to a 0-lattice G its augmented frame of ideals, denoted by *Idl^*G*. The top of Idl^*G is strictly larger than the improper ideal $\bigvee\{\,\downarrow d \mid d \in G\,\} = G$. Idl^*G contains $Idl\,G$ as an open sublocale. If G has no top element, $Idl\,G$ is not compact. In the literature, a frame of the form $Idl\,G$, where G is an object of $\mathbf{D}_0$, is said to be an *algebraic frame with FIP*.

One might also consider distributive lattices possibly without top or bottom and $\vee$-$\wedge$-preserving morphisms. Also, one may consider the frame of ideals of an arbitrary distributive join-semilattice (= an *algebraic frame*).

On σ-Coherent and σ-Algebraic Frames

A *σ-frame* is a set equipped with a countable join operation and a binary meet operation that distributes over countable joins ("Madden-Vermeer-1986.pdf").

- $\sigma\mathbf{Fr}_0^1$ denotes the category of σ-frames with bottom and top element and 0-1-$\bigvee$-$\wedge$-preserving morphisms.
- $\sigma\mathbf{Fr}_0$ denotes the category of σ-frames with bottom element and 0-$\bigvee$-$\wedge$-preserving morphisms.

The left adjoint of the forgetful functor $\mathbf{Frm} \to \sigma\mathbf{Fr}_0^1$ assigns to a 0-1-σ-frame S its frame of σ-ideals (i.e., ideals closed under countable suprema). This is denoted by $Idl_\sigma S$. A frame of the form $Idl_\sigma S$ is said to be *σ-coherent*. The σ-coherent frames are characterized as those whose Lindelöf elements form a generating sub-0-1-σ-frame ("Madden-Vermeer-1986.pdf," Proposition 1.1).

Ad the "unknown representation". The left adjoint of the forgetful functor $\mathbf{Frm} \to \sigma\mathbf{Fr}_0$ assigns to a 0-σ-frame T its augmented frame of σ-ideals, denoted by $Idl_\sigma^* T$. The top of $Idl_\sigma^* S$ is strictly larger than the improper σ-ideal, T. $Idl_\sigma^* T$ contains $Idl_\sigma T$. If T has no top element, $Idl_\sigma T$ is not Lindelöf. (The term, "σ-algebraic with FIP," would fit.)

Regular σ-frames and regular Lindelöf locales

Recall the definitions: Suppose D is a bounded distributive lattice, and $a, b \in D$. We say b is *well-below* a if there is $c \in D$ such that $b \wedge c = 0$ and $a \vee c = 1$. A frame is *regular* if every element is the supremum of the elements that are well-below it. A 0-1-σ-frame is *regular* if every element is the supremum of a countable set of elements that are well-below it.

Lemma. Let S be a 0-1-σ-frame. Then $Idl_\sigma S$ is regular (as a frame) iff S is regular (as a σ-frame). (*Proof.* Exercise.)

Lemma. Suppose $f : A \to B$ is a frame morphism, with A is regular and B Lindelöf. If $a \in A$ is Lindelöf, then so is $f(a)$. (cf. "Madden-1991-kappa.pdf," 4.2)

Proof. There is a countable set X of elements well-below a such that $\bigvee X = a$. For each $x \in X$, select $x' \in A$ such that $a \vee x' = 1_A$ and $x \wedge x' = 0_A$. Suppose $f(a) \leqslant \bigvee Y$ for some $Y \subseteq B$. Since $f(x') \vee \bigvee Y = 1_B$, there is for each $x \in X$, a countable set $Y_x \subseteq Y$ such that $f(x') \vee \bigvee Y_x = 1_B$. Moreover, $f(x) \leqslant \bigvee Y_x$, since $f(x) \wedge f(x') = 0_B$. Thus, $f(a) = \bigvee_{x \in X} f(x) \leqslant \bigvee_{x \in X} \bigvee Y_x = \bigvee(\bigcup_{x \in X} Y_x)$. □

Proposition. The functor Idl_σ is an equivalence between the category of regular 0-1-σ-frames and the category of regular Lindelöf frames. (Madden op. cit., 4.3) □

Some research questions

Question. The definition of regularity requires a top element. Suppose S is a 0-σ-frame with no top element such that for all $s \in S$, the 0-1-σ-frame $\downarrow s$ is regular. Must $Idl_\sigma S$ be regular? (A similar question was asked at the end of Lecture 6.)

Related Question. A distributive lattice L with 1 is said to be *conjunctive* if for all $a, b \in L$: if $a \nleqslant b$, there is $c \in L$ such that $a \vee c = 1$ and $b \vee c \neq 1$. Suppose L is a distributive lattice without top element such that $\downarrow a$ is conjunctive for all $a \in L$. Is $Idl\, L$ conjuctive?

Thoughts toward a general problem. This is an admittedly vague attempt to generalize the two problems above. Suppose that P is a frame property. Say that P is $\bigvee$*-stable* if for any frame F and any family $A \subseteq F$, if $P(\downarrow a)$ for all $a \in A$, then $P(\downarrow \bigvee A)$. We could impose conditions on F or A, e.g., ask about $\bigvee$-stability for directed A in for all frames in some designated class. *How do we recognize* $\bigvee$*-stable properties?*

Example. Suppose F is a frame and let A be a subset of F. If $\downarrow a$ is boolean for all $a \in A$, then $\downarrow \bigvee A$ is boolean.
Proof. Suppose $b \leqslant \bigvee A$. For each $a \in A$, pick c_a such that $(b \wedge a) \vee c_a = a$ and $b \wedge c_a = 0$. Set $c = \bigvee\{ c_a \mid a \in A \}$. Then $b \vee c = \bigvee\{ b \vee c_a \mid a \in A \} = \bigvee A$ (since $a \leqslant b \vee c_a \leqslant \bigvee A$), and $b \wedge c = \bigvee\{ b \wedge c_a \mid a \in A \} = 0$.

Example. With "regular" in place of "boolean," the statement is not true. Let $\mathbb{B}$ be the real line with a new element $0'$ adjoined. The neighborhoods of 0 are the sets containing an interval $(-\epsilon, \epsilon) \subseteq \mathbb{R}$. The neighborhoods of $0'$ are the sets containing some $(-\epsilon, 0) \cup (0, \epsilon) \cup \{0'\}$. Both $\mathbb{B} \backslash \{0\}$ and $\mathbb{B} \backslash \{0'\}$ are regular, but $\mathbb{B}$ is not. Let a be a neighborhood of 0 not containing $0'$, and let b be any neighborhood of 0 contained in a. If $a \cup c = \mathbb{B}$, then $0' \in c$, so c is a neighborhood of $0'$, so $b \cap c \neq \varnothing$.

Localic Yosida: Sketch of proof

1. $\mathcal{Y}(A, e)$ is the frame of archimedean kernels (i.e., relatively-uniformly-closed ℓ-ideals) of A that are contained in the archimedean kernel generated by e. For $a \in A^+$, $y_e(a)$ is the archimedean kernel generated by a.
2. For any $a \in A$, and any rational numbers p and q, define

   $$\Phi_0(a)(p, q) := y_e\big((a - pe)^+ \wedge (qe - a)^+\big).$$

   (Intuitively, this is the open sublocale of $\mathcal{Y}(A, e)$ on which $pe < a < qe$.)
3. Verify that $(p, q) \mapsto \Phi_0(a)(p, q)$ satisfies the *Joyal relations* (see below) for the localic reals $\mathcal{R}$, hence conclude that $\Phi_0(a)$ extends to a frame map $\Phi(a) : \mathcal{R} \to \mathcal{Y}(A, e)$.
4. Verify that $a \mapsto \Phi(a) \in \mathcal{R}\mathcal{Y}(A, e)$ is an ℓ-group homomorphism.
5. Functoriality follows from the nature of the constructions.

The frame of real numbers

Definition. Let $\mathcal{F}$ be a frame. A function $f : \mathbb{Q}^2 \to \mathcal{F}$ is a *Joyal map* if, for all p, q, r, s in $\mathbb{Q}$:

- (R_1) if $q \leqslant p$, then $f(p, q) = \bot$;
- (R_2) $f(p, q) \wedge f(r, s) = f(\max(p, r), \min(q, s))$;
- (R_3) if $p \leqslant r < q \leqslant s$, then $f(p, q) \vee f(r, s) = f(p, s)$;
- (R_4) $\bigvee\{ f(x, y) \mid x, y \in \mathbb{Q} \;\&\; p < x < y < q \} = f(p, q)$;
- (R_5) $\bigvee\{ f(x, y) \mid x, y \in \mathbb{Q} \} = \top$.

The universal Joyal map is denoted by $j : \mathbb{Q}^2 \to \mathcal{R}$. The codomain $\mathcal{R}$ (i.e., the frame freely generated by $\mathbb{Q}^2$ subject to relations (R_1)-(R_5)) is called the *frame of opens of the real numbers* or the *frame of reals* for short. By definition, if $f : \mathbb{Q}^2 \to \mathcal{F}$ is a Joyal map, then there is a unique frame morphism $\overline{f} : \mathcal{R} \to \mathcal{F}$ such that $\overline{f} \circ j = f$.

Localic Yosida

Lemma. (cf. "Madden-1992-frames.pdf" 4.2) Suppose A is an abelian ℓ-group and $e \in A^+$. For each $a \in A$, the map $\Phi_0(a) : \mathbb{Q}^2 \to \mathcal{Y}(A, e)$ defined by:

$$\Phi_0(a)(p, q) := y_e\big((a - pe)^+ \wedge (qe - a)^+\big)$$

satisfies (R_1)–(R_5).

Proof. Suppose $p, q, r, s \in \mathbb{Q}$.

(R_1): Suppose $q \leqslant p$. Then $0 \leqslant (a - pe)^+ \wedge (qe - a)^+ \leqslant (a - pe)^+ \wedge (pe - a)^+ = 0$. So $\Phi_0(a)(p, q) = y_e(0) = \perp$.

(R_2), (R_3): Similar direct computations using the arithmetic of ℓ-groups and referencing (I_1)–(I_4). (R_3) resembles the computation used to prove that $\mathcal{Y}(A, e)$ is regular (Lecture 6, slide 11).

(R_4): $\bigvee\{\, \Phi_0(a)(p', q') \mid p < p' < q' < q \,\} = \bigvee\{\, y_e(a - p'e)^+ \mid p < p' \} \wedge \bigvee\{\, y_e(q'e - a)^+ \mid q' < q \}$. Suppose $q' < q$. Then $(qe \vee a) - (q'e \vee a) \leqslant (q - q')e$, so $(qe - a)^+ - (q'e - a)^+ \leqslant (q - q')e$. Substituting $q' = q - (1/n)$:

$$\big((q - \tfrac{1}{n})e - a\big)^+ \uparrow_e (qe - a)^+$$

.

(R_5): This uses $(e - \frac{1}{n}|a|) \uparrow_{|a|} e$. This is the only place in the proof where we use the full strength of (Y). For (R_4), only e-uniform convergence is needed.