A Course on the Yosida Theorem

Classical & Pointfree Versions & Applications

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Lecture 6. Relative Uniform Convergence & the Yosida Locale

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Definitions

Assume A is an abelian ℓ-group and $a, a_i \in A$ and $s \in A^+$.

Definition. We say, *a is an s-uniform limit of* $\{ a_i \mid i \in \mathbb{N} \}$ or $\{a_i\}$ *converges to a with regulator s* (in symbols, $a_i \to_s a$) if:

$$\forall\, m \in \mathbb{N}\; \exists N_m \in \mathbb{N} \text{ such that } \forall\, i > N_m :\; m|a - a_i| \leqslant s.$$

We write $a_i \uparrow_s a$ to mean $a_i \leqslant a_{i+1}$ for all i and $a_i \to_s a$.

Definition. We say that a is a *relative-uniform limit* (or *r.u.-limit*) of $\{ a_i \,|\! \in \mathbb{N} \}$, if a is an s-uniform limit of $\{ a_i \,|\! \in \mathbb{N} \}$ for some s.

Definition. We say a subset $B \subseteq A$ is *r.u.-closed* if $a \in B$ whenever a is an r.u.-limit of some sequence $\{a_i\} \subseteq B$.

See: A. W. Hager, *Math. Slovaca* 65 (2015), No. 2, 343–358.
(called "Hager-2015b.pdf" in our Dropbox library)

R.U.-Closed ℓ-Ideal = Archimedean Kernel

Exercise. Suppose $I \subseteq A$ is an ℓ-ideal. The following are equivalent:

(i) I is r.u.-closed.

(ii) I^+ is r.u.-closed.

(iii) For any *increasing* sequence $0 \leqslant a_1 \leqslant \cdots$ of elements of I, if a is an r.u.-limit of $\{a_i\}$, then $a \in I$.

Proposition. A/I is archimedean iff I is r.u.-closed

Proof. ($\Rightarrow$) Suppose A/I is archimedean. Given $\{a_i\} \subseteq I^+$, and $a_i \uparrow_s a$, we must show $a \in I$. For each $m \in \mathbb{N}$, $m|a - a_i| \leqslant s$ for large i. Thus, for each $m \in \mathbb{N}$, $m|a + I| \leqslant s + I$. By the archimedean hypothesis, $a + I = 0 + I$, so $a \in I$.

($\Leftarrow$). Suppose A/I is not archimedean. Then, there are $a, s \in A^+ \backslash I$ such that $m(a + I) \leqslant s + I$ for all $m \in \mathbb{N}$. This implies $m|a - 0| = ma \leqslant 2s$ for all $m \in \mathbb{N}$, and hence that that $0 \rightarrow_{2s} a$. Therefore, I is not r.u.-closed. □

Note the synonyms: *r.u.-closed ℓ-ideal = archimedean kernel*

Facts about r.u.-closed ideals

Lemma 1. Suppose A is archimedean. If $a_i \uparrow_s a$ in A, then a is the supremum of $\{ a_i \mid i \in \mathbb{N} \}$.

Proof. If for some i, $a_i \nleqslant a$, then $(a_i - a)^+ > 0$, and for all $j \geqslant i$, $|a_j - a| \geqslant (a_j - a)^+ > (a_i - a)^+$. By the archimedean hypothesis, there is m such that $m(a_i - a)^+ \nleqslant s$, contrary to the assumption that $a_i \uparrow_s a$. Thus, a is an upper bound. If $a_i \leqslant b \nleqslant a$ for all $i \in \mathbb{N}$, then $m(a - b) \nleqslant s$ for some m, so $m(a - a_i) \nleqslant s$ for all i. □

Lemma 2. Suppose $\phi : A \to B$ is an ℓ-group morphism. If $a_i \to_s a$ in A, then $\phi(a_i) \to_{\phi(s)} \phi(a)$ in B.

Proof. If $m|a_i - a| \leqslant s$, then $m|\phi(a_i) - \phi(a)| \leqslant \phi(s)$. □

Comment. Lemma 2 shows that $\mathcal{Y}$ (defined below) is a functor.

An Example and a Comment

Example 1. Let A be the ℓ-group of all (continuous & finitely) piecewise linear $\mathbb{R}$-valued functions on $[0,1]^n$. For any element $a \in A$, let $Z(a)$ denote the zero-set of a. Then $\langle a \rangle = \{\ b \in A \mid Z(a) \subseteq Z(b)\ \}$, since any non-negative PL function that vanishes on $Z(a)$ is bounded above by a multiple of $|a|$. The map $A \to A/\langle a \rangle$ is equivalent to the restriction map $b \mapsto b|_{Z(a)}$. In particular, $\langle a \rangle$ is r.u.-closed (an archimedean kernel). An element $a \in A^+$ is a weak unit iff $Z(a)$ has dimension $< n$.

Comment. R.u.-convergence plays an important role in the theory of archimedean ℓ-groups. Let A be an archimedean ℓ-group. We say *$\{a_i\} \subseteq A$ is s-Cauchy* if

$$\forall m \in \mathbb{N},\ \exists N_m :\ i,j > N_m \Rightarrow m|a_j - a_i| < s.$$

We say A is *r.u.-complete* if for all $s \in A^+$, every s-Cauchy sequence in A has an s-uniform limit in A.

> **Theorem.** *(Veksler-Ball-Hager) The r.u.-complete archimedean ℓ-groups form the strongest essential monoreflective subcategory of* **Arch**. *Moreover, an embedding $f : A \to B$ is isomorphic to the reflection $r_A : A \to rA$ if and only if B is r.u.-complete and f is epic and majorizing.*

ArchK A

The set of all archimedean kernels of A is denoted *ArchK A*.

Fact. *ArchK A* is a complete lattice. For suppose $\{ K_j \mid j \in J \} \subseteq \mathit{ArchK}\ A$. Then $\prod\{ A/K_j \mid j \in J \}$ is archimedean, and $\bigcap\{ K_j \mid j \in J \}$ is the kernel of $a \mapsto (a + K_j) : A \to \prod\{ A/K_j \mid j \in J \}$. We will show that *ArchK A* is a quotient of *Idl A*.

Definition. If $X \subseteq A$, X^* denotes the set of r.u.-limits of sequences in X.

Lemma. If J is an ℓ-ideal, so is J^*.

Proof. J^* is closed under $+$, for if $a_i, b_i \in J$ and $a_i \to_s a$ and $b_i \to_t b$, then $a_i + b_i \in J$, and $(a_i + b_i) \to_{(s+t)} (a + b)$ (since $m|(a + b) - (a_i + b_i)| \leqslant m|a - a_i| + m|b - b_i| \leqslant s + t$), $a_i \wedge h \in J$, so $a + b \in J^*$. Moreover, $(J^*)^+$ is convex, for suppose $a \in J^*$ and $h \in A$ with $0 \leqslant h \leqslant a$. Then $a_i \to_s a$, for some $a_i \in J$ and $s \in A^+$. But then, $a_i \wedge h \in J$ and $(a_i \wedge h) \to_s (a \wedge h) = h$, so $h \in J^*$. □

Lemma. $(J \cap K)^* = J^* \cap K^*$.

Proof. $(\subseteq)$ is clear. $(\supseteq)$ Suppose $f \in J^* \cap K^*$. Then f is an r.u.-limit of elements of J and an r.u.-limit of elements of K. So, $a_i \uparrow_s f$ for some $\{a_i\} \subseteq J$ and $s \in A^+$, and $b_i \uparrow_t f$ for some $\{b_i\} \subseteq K$ and $t \in A^+$. W.l.o.g., $s = t$, since we may replace s and t with $s \vee t$. By definition of s-regulated convergence, there are functions $N, N' : \mathbb{N} \to \mathbb{N}$ such that $m(f - a_i) \leqslant s$ if $i > N(m)$, and $m(f - b_i) \leqslant s$ if $i > N'(m)$. Let $N'' = N \vee N'$. Then

$$m\big((f - a_i) \vee (f - b_i)\big) \leqslant s, \text{ if } i > N''(m),$$

but $(f - a_i) \vee (f - b_i) = f + (-a_i \vee -b_i) = f - (a_i \wedge b_i)$. Since $a_i \wedge b_i \in J \cap K$, $f \in (J \cap K)^*$. □

ArchK A is a quotient of *Idl A*

Starting with any $J \in \mathit{Idl}\,A$, if we iterate the $*$-operation transfinitely, we eventually reach a stable value. More precisely, let $J^0 = J$. For ordinals α, let $J^{\alpha+1} := (J^\alpha)^*$. For limit ordinals λ, let $J^\lambda := \bigcup\{\, J^\alpha \mid \alpha < \lambda \,\}$. Let $\mathit{ru}(J) := J^{\omega_1}$.

Proposition. $\mathit{ru} : \mathit{Idl}\,A \to \mathit{Idl}\,A$ is a nucleus, whose image is *ArchK A*.

Proof. By construction $J \subseteq \mathit{ru}(J) = \mathit{ru}(\mathit{ru}(J))$. By the Lemma, $\mathit{ru}(J \cap K) = \mathit{ru}(J) \cap \mathit{ru}(K)$, so ru is a nucleus. Evidently, $J \in \mathit{ArchK}\,A$ iff $J = J^*$ iff $J = \mathit{ru}(J)$. □

Problem. Give "nice" (e.g., finite, or easy to check, and useful) conditions on $a, b \in A$ for $\mathit{ru}(\langle a \rangle) = \mathit{ru}(\langle b \rangle)$ and for $\langle a \rangle = \mathit{ru}(\langle a \rangle)$.

The Yosida Frame $\mathcal{Y}A$ of an ℓ-group A

Definition. Let A be an abelian ℓ-group, and let $\mathcal{G}$ be a frame. We say that $g : A^+ \to \mathcal{G}$ is a *Yosida map out of A* if for all $a, b \in A^+$:

- (I_1) $g(0_A) = \bot$,
- (I_2) $g(a \wedge b) = g(a) \wedge g(b)$,
- (I_3) $g(a \vee b) = g(a) \vee g(b)$,
- (I_4) $g(a + b) = g(a) \vee g(b)$;
- (Y) if $\{a_i\}_{i=0}^{\infty} \subseteq A^+$ and $a_i \uparrow_b a$, then $g(a) = \bigvee_{i=1}^{\infty} g(a_i)$.

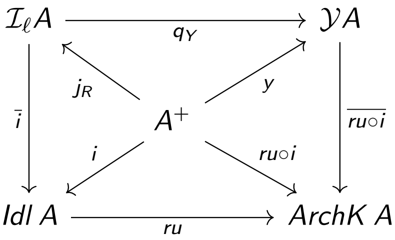
The *universal Yosida map out of A* is denoted $y : A^+ \to \mathcal{Y}A$.

Remarks. In the terminology of the last lecture, if we let $R = R_{I\&Y}(A^+)$, then $\mathcal{Y}A = \mathcal{F}A^+/R$ and $y = j_R$. In the definition above, we have used a more efficient way of speaking about *frames by generators and relations*.

Relationships between $\mathcal{Y}A$ and $ArchK\ A$

In the following diagram:

- q_Y is the frame quotient mapping induced by the relation Y;
- j_R and y are the "presentation maps" for $\mathcal{I}_\ell A$ and $\mathcal{Y}A$;
- i takes $a \in A^+$ to the ℓ-ideal $\langle a \rangle$;
- ru is the nucleus described on the previous slide, viewed as a frame morphism;
- $\bar{i}$ and $\overline{ru \circ i}$ are the induced frame morphisms.



If $\mathcal{G}$ is a frame and we adjoin a new top element that is different from $\top_{\mathcal{G}}$, we call the result the *augmentation of $\mathcal{G}$*. $\mathcal{I}_\ell A$ is the augmentation of $Idl\ A$, and $\mathcal{Y}A$ is the augmentation of $ArchK\ A$.

$\mathcal{Y}(A, e)$

Definition. Suppose $e \in A^+$. $\mathcal{Y}(A, e)$ denotes the quotient of *ArchK A* (or of $\mathcal{Y}A$) obtained by identifying $y(e)$ with the top. (This is the open sublocale of $\mathcal{Y}A$ corresponding to the map $z \mapsto z \wedge y(e)$.)

Theorem. If A is divisible, $\mathcal{Y}(A, e)$ is regular.

Remark. $y_e\big((a-b)^+\big) = [\![a > b]\!] =$ "the extent to which $a > b$".

Proof. Let $y_e(a) := y(a) \wedge y(e)$. The elements $y_e(a)$, $a \in A^+$ generate $\mathcal{Y}(A, e)$. By relation Y,

$$y_e(a) = \bigvee \{\, y_e\big((a - \tfrac{1}{n}e)^+\big) \mid n = 1, 2, \ldots \},$$

since $|a - (a - \frac{1}{n}e)^+| \leqslant \frac{1}{n}e$. Suppose $1 > p > s > q > 0$ in $\mathbb{Q}$. Then

$$\begin{aligned} y_e\big((a-qe)^+\big) \vee y_e\big((se-a)^+\big) &= y_e\big((a-qe) \vee (se-a) \vee 0\big) \\ &= y_e\Big(\big[(a - \tfrac{q+s}{2}e) \vee (\tfrac{q+s}{2}e - a)\big] + \tfrac{s-q}{2}e \vee 0\Big) \\ &= y_e\Big(\big|(a - \tfrac{q+s}{2}e)\big| + \tfrac{s-q}{2}e \vee 0\Big) = y_e(e) = \top \end{aligned}$$

By a similar argument, $y_e\big((a-pe)^+\big) \wedge y_e\big((se-a)^+\big) = \bot$. Thus $y_e\big((a-pe)^+\big)$ is well-inside $y_e\big((a-qe)^+\big) \leqslant y_e(a)$. □

Comments. (1) The proof actually demonstrates completely regularity. (2) The divisiblity hypothesis can be dispensed with. (3) As a corollary, *ArchK A* is locally regular. *I do not know if it is the case that ArchK A is regular for all A.*

What's next?

We have developed two representation theorems:

(I) the embedding of an arbitrary abelian ℓ-group in a product of totally-ordered groups; and

(II) the embedding of an arbitrary archimedean ℓ-group with weak unit in a $D(X)$, X a compact T_2-space.

We are in the middle of developing a third:

(III) an embedding of an arbitrary archimedean ℓ-group with weak unit in a $C(L)$, L a regular Lindelöf locale.

We have nearly finished understanding the "representation space" $L = \mathcal{Y}(A, e)$ of (III). We have yet to show that it is Lindelöf. After this, we will equip this space with an ℓ-group of functions, and then embed A in this ℓ-group.

Ultimate goal. We conjectured that there is a fourth representation theorem that: (i) applies to arbitrary archimedean ℓ-groups, (ii) generalizes the localic represenation, (iii) is functorial, and (iv) enables us to deduce much of the known theory of the category **Arch** of archimedean ℓ-groups. The goal of this course is to discover this as-yet unknown representation.