A Course on the Yosida Theorem

Classical & Pointfree Versions & Applications

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Lecture 5. Frames and Locales

Frames by Generators and Relations

The spectrum of an ℓ-group

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Semilattices

Definition. A semilattice L is a set equipped with an associative, commutative, idempotent binary operation (i.e., an idempotent, commutative semigroup).

Some notational conventions. If the operation is denoted $\wedge$ (meet), we call L a *$\wedge$-semilattice*. In this case, we write $a \leqslant b$ if $a \wedge b = a$, $a, b \in L$. As we proved earlier, this makes L into a poset in which $a \wedge b$ is the greatest lower bound of a and b.

We do not require a semilattice to have an identity. If there is one, it is unique. An element of a $\wedge$-semilattice L is an identity iff it is the largest element of L.

An element z in a $*$-semigroup L is said to be a *zero for $*$*, or an *absorbing element for $*$* if $z * a = z = a * z$ for all $a \in L$. A semigroup can have at most one zero. A zero in a $\wedge$-semilattice (if there is one) is the smallest element of L.

Free semilattices

Proposition. For any set X, let $\mathcal{S}X$ denote the set of non-empty finite subsets of X with the operation $*$ defined by $A * B := A \cup B$. Then $\mathcal{S}X$ is the *free $*$-semilattice on X*.

If we include the empty set, then we get an identity and the free 1-$*$-semilattice.

Suppose the operation is $\wedge$. The order induced on $\mathcal{S}X$ by $\wedge$ is the opposite of the containment order: $A \leqslant B \iff B \subseteq A$.

Proof Sketch. Suppose L is a $*$-semilattice and $\phi : X \to L$ is a set map. We must show that ϕ has a unique extension to an $*$-morphism from $\mathcal{S}X$ to L. Define $\overline{\phi} : \mathcal{S}X \to L$ by $\overline{\phi}(A) := \circledast\{\, \phi(a) \mid a \in A \,\}$. Then,

$$\begin{aligned}\overline{\phi}(A * B) &= \circledast\{\, \phi(x) \mid x \in A \cup B \,\} \\ &= \Big(\circledast\{\, \phi(x) \mid x \in A \,\}\Big) * \Big(\circledast\{\, \phi(x) \mid x \in B \,\}\Big) \\ &= \overline{\phi}(A) * \overline{\phi}(B). \qquad \square\end{aligned}$$

Frames

Definition. A *frame* $\mathcal{A}$ is a $\wedge$-semilattice equipped with distinguished elements $\top$ and $\bot$, and a map $\bigvee$ from the power set of $\mathcal{A}$ to $\mathcal{A}$ such that:

(*i*) for any $a \in \mathcal{A}$, $\bot \leqslant a \leqslant \top$;

(*ii*) for any $B \subseteq \mathcal{A}$ and any $b \in B$, $b \leqslant \bigvee B$;

(*iii*) for any $a \in \mathcal{A}$ and any $B \subseteq \mathcal{A}$, $a \wedge \bigvee B = \bigvee\{\, a \wedge b \mid b \in B \,\}$.

A *frame morphism* is a function between frames that preserves the frame operations $\top$, $\bot$, $\wedge$ and $\bigvee$.

Here, we have made an "equational" definition of a frame: a frame is a set with operations that obey certain equational laws. (The inequalities can be stated as equations using $\wedge$.)

$\bigvee B$ is the least upper bound of B in the order induced by $\wedge$, for if $b \leqslant u \in \mathcal{F}$ for all $b \in B$, then $\bigvee B = \bigvee\{\, u \wedge b \mid b \in B \,\} = u \wedge \bigvee B$, so $\bigvee B \leqslant u$.

Free Frames

Definition. If L is a poset, $\mathcal{D}L$ denotes the set of all down-sets of L. For any $a \in L$, $\downarrow a = \{\, b \in L \mid b \leqslant a \,\} \in \mathcal{D}L$. $\mathcal{E}L$ denotes $\mathcal{D}L$ with a top element $\top_{\mathcal{E}L}$ adjoined.

Note that $\mathcal{D}L$ has operations $\wedge :=$ binary intersection and $\bigvee :=$ arbitrary union, and $\wedge$ distributes over $\bigvee$, because these are simply set-theoretic operations. The operations extend uniquely to $\mathcal{E}L$, making it a frame.

The empty set is the bottom element or $\mathcal{D}L$. The top element of $\mathcal{D}L$ is $L = \bigvee\{\, \downarrow a \mid a \in L \,\}$, but this is different from $\top_{\mathcal{E}L}$.

Theorem. For any set X, let $\mathcal{F}X := \mathcal{E}(\mathcal{S}X)$, and let $j : X \to \mathcal{F}X$ be the map defined by $j(x) := \downarrow\{x\}$. Then $(\mathcal{F}, j)$ is the free frame on X, i.e., if $\mathcal{A}$ is any frame and $\phi : X \to \mathcal{A}$ is any set map, then there is a unique frame morphism $\overline{\phi} : \mathcal{F}X \to \mathcal{A}$ such that $\phi = \overline{\phi} \circ j$.

The Theorem follows immediately from the following:

Lemma. Suppose L is a $\wedge$-semilattice. Then $\mathcal{E}L$ is the free frame on L, i.e., any $\wedge$-preserving map $\phi : L \to \mathcal{A}$, where $\mathcal{A}$ is a frame, has a unique extension to a frame map $\overline{\phi} : \mathcal{E}L \to \mathcal{A}$.

Proof Sketch. Let $\overline{\phi}(D) := \bigvee\{\, \phi(a) \mid a \in D \,\}$, for any $D \in \mathcal{D}L$, and let $\overline{\phi}(\top_{\mathcal{E}L}) = \top_{\mathcal{A}}$. (See Johnstone, *Stone Spaces* II.1.2. The most interesting part of the proof is the verification that $\overline{\phi}$ preserves $\wedge$.)

Frame Congruence Relations

Definition. Suppose $\mathcal{A}$ is a frame and R is an equivalence relation on $\mathcal{A}$. We say that R is a *frame congruence relation* if it "respects the operations," i.e., if $a_i, a'_i \in \mathcal{A}$ and $a_i \, R \, a'_i$ for all $i \in I$, then $(a_0 \wedge a_1) \, R \, (a'_0 \wedge a'_1)$ and $\bigvee\{ a_i \mid i \in I \} \, R \, \bigvee\{ a'_i \mid i \in I \}$.

Facts.

- $R \subseteq \mathcal{A} \times \mathcal{A}$ is a frame congruence relation on $\mathcal{A}$ iff R is an equivalence relation and a sub-frame of $A \times \mathcal{A}$.
- Any intersection of frame congruence relations on $\mathcal{A}$ is a frame congruence relation on $\mathcal{A}$.
- Given any relation on $\mathcal{A}$, there is a smallest congruence relation containing it.

Frames by Generators and Relations

Every element of $\mathcal{F}X$ can be written in the form $\bigvee B$, where each element of B is of the form $x_1 \wedge \cdots \wedge x_n$ for some finite set $\{ x_1, \ldots, x_n \} \subseteq X$. We call such an expression a *frame word in X*.

Example. $\bigvee\{ x_{i1} \wedge \cdots \wedge x_{in_i} \mid i \in I \}$

Suppose X is a set and R is a set of equations between frame words in X. Let $\mathcal{F}X/R$ denote the quotient of $\mathcal{F}X$ by the smallest frame congruence containing R. Let $j_R : X \to \mathcal{F}X/R$ denote the composition of set map $j : X \to \mathcal{F}X$ followed by the quotient map $\mathcal{F}X \to \mathcal{F}X/R$.

Frames by Generators and Relations

Fact. Suppose $\mathcal{A}$ is a frame and $\phi : X \to \mathcal{A}$ is a set map. Suppose further that $\overline{\phi}(w_1) = \overline{\phi}(w_2)$ for all equations $w_1 = w_2$ in R. Then, the the kernel of $\overline{\phi}$ contains R, so (by the isomorphism theorem) there is a unique frame morphism $\tilde{\phi} : \mathcal{F}X/R \to \mathcal{A}$ such that $\tilde{\phi} \circ j_R = \phi$.

$$
\begin{tikzcd}
X \arrow[r, "j"'] \arrow[rr, bend left, "j_R"] \arrow[dr, "\phi"'] & \mathcal{F}X \arrow[r] \arrow[d, "\overline{\phi}"] & \mathcal{F}X/R \arrow[dl, "\tilde{\phi}"] \\
& \mathcal{G} &
\end{tikzcd}
$$

Localic ℓ-spectrum

For any abelian ℓ-group A, let $R_\ell(A^+)$ be the set of equations (in $\mathcal{F}A^+$) of the following form, where $0, a, b \in A^+$:

(l_1) $j(0) = \bot$,
(l_2) $j(a \wedge b) = j(a) \wedge j(b)$,
(l_3) $j(a + b) = j(a) \vee j(b)$,
(l_4) $j(a \vee b) = j(a) \vee j(b)$.

Note that these equations are not true in $\mathcal{F}A^+$, but if we write j_R in place of j, then we have true equations in $\mathcal{F}A^+/R_\ell(A^+)$

Definition. The *localic spectrum* of an ℓ-group A is

$$\mathcal{I}_\ell A := \mathcal{F}A^+/R_\ell(A^+).$$

Fact. For $a \in A^+$, let $i(a) := \langle a \rangle$. The map i satisfies the relations l_1–l_4, so we have a frame morphism $\bar{i} : \mathcal{I}_\ell A \to \mathit{Idl}\, A$. On the next slide, we examine this map.

$\bar{i} : \mathcal{I}_\ell A \to \mathit{Idl}\, A$

I_1	$j(0) = \bot$	$\langle 0 \rangle \subseteq I$ for all $I \in \mathit{Idl}\, A$
I_2	$j(a \wedge b) = j(a) \wedge j(b)$	$\langle a \wedge b \rangle = \langle a \rangle \cap \langle b \rangle$
I_3	$j(a + b) = j(a) \vee j(b)$	$\langle a + b \rangle = \langle a \rangle \vee \langle b \rangle$
I_4	$j(a \vee b) = j(a) \vee j(b)$	$\langle a \vee b \rangle = \langle a \rangle \vee \langle b \rangle$

Proposition. The map $\bar{i} : \mathcal{I}_\ell A \to \mathit{Idl}\, A$ is surjective and injective on $\mathcal{I}_\ell A \backslash \top_{I_\ell A}$.

Proof. Surjectivity is obvious. By I_2, every element of $\mathcal{I}_\ell A$ other than the top can be written in the form $\bigvee \{ j_R(g) \mid g \in G \}$ for some subset $G \subseteq A^+$. Suppose $\langle G \rangle = \langle H \rangle$ for some $H \subseteq A^+$. We must show that $\bigvee \{ j_R(g) \mid g \in G \} = \bigvee \{ j_R(h) \mid h \in H \}$. For any $h \in H$, there is a finite list $g_1, \ldots, g_n$ of elements of G such that $h \leqslant g_1 + \cdots + g_n$. But then $j_R(h) \leqslant \bigvee \{ j_R(g) \mid g \in G \}$. Since h was an arbitrary element of H, $\bigvee \{ j_R(h) \mid h \in H \} \leqslant \bigvee \{ j_R(g) \mid g \in G \}$. The desired equality follows by symmetry. $\square$

Note: The map $\bar{i}$ must take the top element of $\mathcal{I}_\ell A$ to the top of $\mathit{Idl}\, A$, which is $\langle A^+ \rangle$. Thus, while $\top_{\mathcal{I}_\ell A} > \bigvee \{ j_R(a) \mid a \in A^+ \}$ in $\mathcal{I}_\ell A$, $\bar{i}$ takes both these elements to $\langle A^+ \rangle$. These are the only two elements of $\mathcal{I}_\ell A$ that are identified by $\bar{i}$.

$\mathcal{I}_\ell$ is a functor

Suppose $\phi : A \to B$ is a morphism of abelian ℓ-groups. Then

$$j_R \circ \phi : A^+ \to \mathcal{I}_\ell B$$

satisfies I_1–I_4. I_2, for example, is verified as follows ($a, b \in A$):

$$(j_R \circ \phi)(a \wedge b) = j_R(\phi(a) \wedge \phi(b)) = (j_R \circ \phi)(a) \wedge (j_R \circ \phi)(b).$$

Thus, there is a unique frame morphism $\mathcal{I}_\ell\phi : \mathcal{I}_\ell A \to \mathcal{I}_\ell B$ such that $(\mathcal{I}_\ell) \circ j_R = \phi \circ j_R$.

$$\begin{tikzcd} A^+ \arrow[r, "\phi"] \arrow[d, "j_R"'] & B^+ \arrow[d, "j_R"] \\ \mathcal{I}_\ell A \arrow[r, "\mathcal{I}_\ell \phi"] & \mathcal{I}_\ell B \end{tikzcd}$$

Why isn't $\bigvee\{\, j_R(a) \mid a \in A \,\}$ the top of $\mathcal{I}_\ell A$?

Answer: Functoriality! $\phi(A)$ may generate a proper idea of B, in which case,

$$\mathcal{I}_\ell\phi\Big(\bigvee\{\, j_R(a) \mid a \in A \,\}\Big)$$

is not the top of $\mathcal{I}_\ell B$. Thus, while *Idl* A is a frame for any abelian ℓ-group A, *Idl* is *not* functorial. $\mathcal{I}_\ell$ repairs this.

$\mathcal{I}_\ell A$ always has a frame morphism to $\{\bot, \top\}$ that sends $\top_{\mathcal{I}_\ell A}$ to $\top$ and all other elements of $\mathcal{I}_\ell A$ to $\bot$.

Viewing $\mathcal{I}_\ell A$ as a locale, this morphism is a closed point whose only open neighborhood is the whole locale. Suppose X and Y are topological spaces and Y contains such a point q. Let U be a proper open subset of X, and let $f : X \to Y$ be continuous on U and satisfy $f(x) = q$ iff $x \in X \backslash U$. For any open V in Y, either $q \notin V$ and $f^{-1}(V)$ is an open subset of U, or $q \in V$ and $f^{-1}(V) = X$. Thus, a function from X to Y is the same thing as an open subset of X and a continuous function from that set to $Y \backslash \{q\}$.

What's Next?

Archimedean kernels (relatively-uniformly-closed ideals)

Localic real numbers

Localic Yosida

Here is a link to a good lecture by Anrdé Joyal on frames and locales (from "A crash course in topos theory: the big picture") https://youtu.be/Ro8KoFFdtS4