A Course on the Yosida Theorem

Classical & Pointfree Versions & Applications

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Lecture 4. Applications:
Archimedean ℓ-Groups with Strong Unit
Monoreflections

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The Yosida Representation (review of last lecture)

An important and influential reference for the material in Lectures 2 & 3 (and more) is:

Hager A W & Robertson L C, Representing and ringifying a Riesz space, *Symposia Mathematica* 21 (1975), 411 – 431.

A copy is in the Dropbox folder ("Hager-Robertson-1975.pdf")

In the present lecture, we will review the Yosida Representation, and then apply it to the category $\mathbf{W}^*$ of archimedean ℓ-groups with strong unit. A major goal of much research over the past several decades has been to generalize results concerning $\mathbf{W}^*$ to the categories $\mathbf{W}$ (archimedean ℓ-groups with weak unit) and **Arch** (archimedean ℓ-groups in general). In this lecture, we will present some of the $\mathbf{W}^*$ prototype theorems.

Recall Yosida Part 1

Suppose A is an archimedean ℓ-group with weak unit u. Then there is a compact Hausdorff space $Y(A,u)$ and an ℓ-isomorphism:

$$\widehat{(\ )}_u : A \to \widehat{A}_u \subseteq D(Y(A,u)),$$

where $\widehat{A}_u$ is a point-separating* ℓ-group of continuous extended-real-valued functions on $Y(A,u)$ and $\widehat{u}_u = 1$.

* We say a set A of functions in $D(X)$ is point-separating if for any $x_0, x_1 \in X$, there is $a \in A$ such that $a(x_0) = 0$ and $a(x_1) \neq 0$.

Recall Yosida Part 2

Suppose B is an archimedean ℓ-group with weak unit v, and $\phi : A \to B$ is an ℓ-group morphism with $\phi(u) = v$.

Then there is a continuous map: $Y(\phi) : Y(B, v) \to Y(A, u)$ such that

$$\widehat{\phi(a)}_v = \widehat{a}_u \circ Y(\phi).$$

Recall Yosida Part 3

Suppose A is a point-separating ℓ-group of continuous extended-real-valued functions on a compact Hausdorff space X, and $1 \in A$.

Then $x \mapsto M_x := \{ a \in A \mid a(x) = 0 \}$ is a homeomorphism of X with $Y(A, 1)$, and $\widehat{a}_1(M_x) = a(x)$ for all $a \in A$.

The Categories $\mathbf{W}$ and $\widehat{\mathbf{W}}$

Objects of $\mathbf{W}$: Pairs (A, u), A an archimedean ℓ-group; $u \in A$ a weak unit.

Morphisms of $\mathbf{W}$: ℓ-group morphisms $\phi : (A, u) \to (A', u')$, with $\phi(u) = u'$.

Objects of $\widehat{\mathbf{W}}$: Pairs (A, X), where X is a compact Hausdorff space and A is a point-separating ℓ-group of continuous extended-real-valued functions on X containing 1.

Morphisms of $\widehat{\mathbf{W}}$: A morphism from (A, X) to (A', X') is a continuous map $\Phi : X' \to X$ such that for all $a \in A$, $a \circ \Phi \in A'$.

This implies in particular that $\Phi^{-1}(a^{-1}(+\infty))$ must be nowhere dense in X' for all $a \in A$; see (Hager Robertson 1975, Remark 2.13). A, as an algebra of functions on X, endows X with a geometric structure. The condition $a \circ \Phi \in A'$ is a demand that Φ preserve this structure. This is an embryonic version of idea underlying the idea of a "structure sheaf" — a topic that we will explore in detail later.

Yosida Theorem: Categorical Interpretation

The Yosida Theorem says that the categories $\mathbf{W}$ and $\widehat{\mathbf{W}}$ are equivalent, via the functor $\mathbf{Y}$, where

$$\mathbf{Y}(A,u) := (\widehat{A}_u, Y(A,u))$$
$$\mathbf{Y}(\phi) := Y(\phi)$$

Because of this, we need not distinguish between the two categories. Given any $\phi : (A,u) \to (B,v)$ in $\mathbf{W}$, we may assume without any loss of generality that

- A and B are ℓ-groups of continuous extended-real-valued functions on spaces $Y(A,u)$ and $Y(B,v)$ and
- ϕ is induced by a continuous map $f : Y(B,v) \to Y(A,u)$.

Strong Units: the Categories $\mathbf{W}^*$ and $\mathbf{C}^*$

Definition. Let A be an ℓ-group. An element $e \in A$ is called a *strong unit* if $0 \leq e$ and for every $a \in A$, there is $n \in \mathbb{N}$ such that $|a| \leq n\, e$.

Definition.

(*i*) The category $\mathbf{W}^*$ is the full subcategory of $\mathbf{W}$ whose objects are those pairs (A, e) such that e is a strong unit.

(*ii*) The category $\mathbf{C}^*$ is the full subcategory of $\mathbf{W}^*$ whose objects are those pairs (A, e) such that $\widehat{A}_e = C^*(Y(A, e))$ (all continuous real-valued functions on $Y(A, e)$—necessarily bounded, since $Y(A, e)$ is compact).

$\mathbf{C}^*$ is a monoreflective subcategory of $\mathbf{W}^*$

Suppose (A, e) is an object of $\mathbf{W}^*$. Then we have a $\mathbf{W}^*$ morphism

$$\rho_A : A \to \rho A := C^*(Y(A, e)), \text{ where } \rho_A(a) := \widehat{a}_e.$$

Comment: We have merely renamed a few things that we've already encountered — ρ_A is the representation morphism $\widehat{(\ )}_e$. We do this to be able say things in a manner that meshes with category theory. In particular, we will show that ρ_A is a reflection map—a concept we will define on the next slide.

Theorem. Let (B, d) be an object of $\mathbf{C}^*$, and let $\phi : (A, e) \to (B, d)$ be a morphism of $\mathbf{W}^*$. Then there is a $\mathbf{C}^*$-morphism $\overline{\phi} : (\rho A, e) \to (B, d)$ such that $\phi = \overline{\phi} \circ \rho_A$.

Proof. Without loss of generality, we may assume $B = C^*(Y(B, d))$. ϕ is induced by a continuous map $f : Y(B, d) \to Y(A, e)$, such that $\widehat{\phi(a)}_d = \widehat{a}_e \circ f$. Let c be any continuous function on $Y(A, e)$. Since $\widehat{c}_e \circ f$ is a continuous, real-valued function on $Y(B, d)$, we have $\widehat{c}_e \circ f \in B$. Thus, we may define $\overline{\phi}$ by $\overline{\phi}(c) = \widehat{c}_e \circ f$, for $c \in \rho A$. □

Reflective subcategories

Definition. Suppose **C** is a category. We say **R** is *reflective* in **C** if:

- **R** is a full, isomorphism- closed subcategory of **C**, and
- for each object A of **C**, there is a morphism $\rho_A : A \to \rho A$ that is universal to **R**, i.e., ρA is in **R** and for any **C**-morphism $\phi : A \to B$ with codomain B in **R**, there is a **unique** morphism $\overline{\phi} : \rho A \to B$, with $\phi = \overline{\phi} \circ \rho_A$.

$$\begin{tikzcd} A \arrow[r, "\rho_A"] \arrow[dr, "\phi"'] & \rho A \arrow[d, dashed, "\exists!\,\overline{\phi}"] \\ & B \end{tikzcd}$$

If ρ_A is monic (i.e., left-cancellable) for all A, then **R** is said to be *monoreflective*. Some examples of monoreflective subcategories:

C	Tych. sp.	distr. latt.	torsion-free ab. grps	$\mathbf{W}^*$
R	cpt T_2 sp.	bool. latt.	divisible t-f ab. grps	$\mathbf{C}^*$

Free objects in $\mathbf{W}^*$

Definition. Let $I := [0,1] \subseteq \mathbb{R}$. Let S be a set. For $\alpha \in S$, let $x_\alpha : I^S \to I$ be the projection onto the α^{th} factor, and let $F_\ell^*(S)$ denote the sub-ℓ-group of $C(I^S)$ generated by $\{\, x_\alpha \mid \alpha \in S \,\} \cup \{1\}$.

Each element of $F_\ell^*(S)$ is of the form $\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n_i} f_{ij}$, where $m, n_i \in \mathbb{N}$ and each f_{ij} is an integer plus an integer-linear combination of finitely many elements of $\{\, x_\alpha \mid \alpha \in S \,\} \cup \{1\}$.

Proposition. Suppose $(A, e) \in \mathbf{W}^*$ and $\alpha \mapsto a_\alpha$ is a function from S to A, with $0 \le a_\alpha \le e$ for all $\alpha \in E$. Then, there is a unique ℓ-group morphism $\phi : F_\ell^*(S) \to A$ such that $\phi(1) = e$ and $\phi(x_\alpha) = a_\alpha$ for all $\alpha \in S$.

Proof. For each $\alpha \in S$, $\widehat{a_\alpha}_e : Y(A, e) \to I$. Consider the map $\Phi : Y(A, e) \to I^S$ whose component at α is $\widehat{a_\alpha}_e$. For all $f \in F_\ell^*(S)$ define $\phi(f) := f \circ \Phi$. Clearly, $x_\alpha \circ \Phi = \widehat{a_\alpha}_e$. Moreover, if $f, g \in F_\ell^*(S)$, then $(f \vee g) \circ \Phi = (f \circ \Phi) \vee (g \circ \Phi)$ and $(f + g) \circ \Phi = (f \circ \Phi) + (g \circ \Phi)$, so ϕ is an ℓ-group morphism. $\square$

Epimorphisms in $\mathbf{W}^*$

Definition. A morphism $\epsilon : A \to B$ in a category $\mathbf{C}$ is said to be $\mathbf{C}$-*epi* if for all $\mathbf{C}$-morphisms $\phi, \psi : B \to C$, $\phi \circ \epsilon = \psi \circ \epsilon$ implies $\phi = \psi$.

Examples. **Set**-epi = surjective. A morphism $f : A \to B$ of bounded distributive lattices is epi if every element of B is either in $f(A)$ or is the complement of an element of $f(A)$.

Proposition. A $\mathbf{W}^*$-morphism $\epsilon : (A, a) \to (B, b)$ is epi iff $Y(\epsilon) : Y(B, b) \to Y(A, a)$ is injective.

Proof Sketch. ϵ is $\mathbf{W}^*$-epi iff $\rho\epsilon$ is $\mathbf{C}^*$-epi iff $Y(\rho\epsilon)$ is **CptHaus**-mono iff $Y(\rho\epsilon) = Y(\epsilon)$ is injective. $\square$

Cor. An object (A, a) of $\mathbf{W}^*$ is *epicomplete* iff it is a $C^*(X)$ for some compact Hausdorff space X.

Some guiding problems suggested by **W***

In the lectures that follow, our goal will be to develop the theory needed to address the following:

- Is there a functorial representation for **Arch**? (Unsolved.)
- What are the free objects in **W**? (Solved, and easy to represent.) In **Arch**? (Solved, but not well-understood; seldom applied.)
- What are the epimorphisms in **W**? In **Arch**? (Solved by Ball & Hager. Not easy.)
- What are the epi-closed objects in **W**? In **Arch**? (Solved for **W** by Madden & Vermeer using locales. Solved for **W** and **Arch** by Ball & Hager without locales. Not easy.)
- What can we say about the monoreflective subcategories of **W**? (Much is known.) Of **Arch**? (Important problems are unsolved.)