A Course on the Yosida Theorem

Classical & Pointfree Versions & Applications

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Lecture 3. The Yosida Representation

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Recall

Throughout, A is an abelian ℓ-group and $a \in A^+$.

$Y(A, a)$ is the set of all ℓ-ideals of A that are maximal missing a, with the topology generated by all $\operatorname{coz}_a b$, $b \in A$, where

$\operatorname{coz}_a b := \{\, M \in Y(A, a) \mid b \notin M \,\}$.

Thm. $Y(A, a)$ is compact and Hausdorff.

Lem. If $X \subseteq Y(A, a)$, $cl\, X = \{\, M \mid \bigcap X \subseteq M \,\}$.

$\widehat{b}_a : Y(A, a) \to \mathbb{R} \cup \{\pm\infty\}$

If $b \in A^+$, $\widehat{b}_a(M) = \sup\{\, m/n \mid ma + M \leq nb + M,\ m, n \in \mathbb{N}, n \neq 0 \,\}$.

Prop. $\widehat{b}_a$ is continuous, for all $b \in A$.

$$\operatorname{fin}_a b := \{\, M \in Y(A, a) \mid b \in \langle M, a \rangle \,\} = (\widehat{b}_a)^{-1}(\mathbb{R})$$

Further comments

(1) Note that all the definitions on the previous page make sense even when A is not archimedean.

(2) The function

$$\widehat{(\ )}_a : b \mapsto \widehat{b}_a : A \to (\mathbb{R} \cup \{\pm\infty\})^{Y(A,a)}$$

preserves $\vee$ and $-$:

$$\widehat{(b \vee c)}_a = \widehat{b}_a \vee \widehat{c}_a;$$
$$\widehat{(-b)}_a = -(\widehat{b}_a).$$

However, $\infty + (-\infty)$ is not defined, so $\widehat{b}_a + \widehat{c}_a$ may not be meaningful.

(3) When A is archimedean, the problem in (2) can be avoided, as we now show.

Maximal ℓ-ideals in Archimedean case

Definition. For $a \in A$, $a^\perp := \{\, b \in A \mid 0 = |a| \wedge |b| \,\}$. We say a is a *weak unit* if $0 \le a$ and $a^\perp = \{0\}$.

Theorem. Suppose A is archimedean, $a, b \in A$ and $0 \le a \le b$. Then $a^\perp$ is the intersection of the values M of a such that $b \in M^*$.

Corollary. If A is archimedean, then $a^\perp$ is the intersection of the values of a, and $\mathrm{fin}_u\, b$ is dense in $Y(A, a)$ for all $b \in A$.

Proof. If $a \notin M$ then $a^\perp \subseteq M$, so $a^\perp \subseteq \bigcap Val(A, a)$. To prove the opposite inclusion, suppose $0 \le x \notin a^\perp$. Then, $0 < x \wedge a$. Let $d := x \wedge a$. Note that $d \le b$. Since A is archimedean and $0 < d$, we may—and do— pick $n \in \mathbb{N}$ such that $n\,d \not\le b$. Let

$$h := b - (n\,d \wedge b), \text{ and } g := n\,d - (n\,d \wedge b).$$

Note that $g \wedge h = 0$. Pick P maximal missing g. Since P is prime, $h \in P$. Also, P does not contain b (otherwise, it would contain g, because $0 < n\,d \le n\,b$ and $(n\,d \wedge b) \le b$). Enlarge P to a value M of b. Since $b \notin M$ but $h \in M$, $n\,d \wedge b \notin M$, so $d \notin M$, so neither x nor a is in M. Clearly $a \in \langle M, b \rangle = M^*$, so M is a value of a. □

Aside: an observation

Let us define a new operation $a \mathbin{\backslash\!\backslash} b := a - (a \wedge b)$.

Note that $(a \mathbin{\backslash\!\backslash} b) \wedge (b \mathbin{\backslash\!\backslash} a) = 0$.

This operation has appeared previously in several contexts:

- $a \mathbin{\backslash\!\backslash} 0 = a - (a \wedge 0) = a + (-a \vee 0) = a^+$
- $0 \mathbin{\backslash\!\backslash} a = 0 - (0 \wedge a) = 0 + (0 \vee -a) = a^-$
- In proving $Y(A, a)$ Hausdorff (to find disjoint cozero nbhds)
- In proving the last theorem, we used $b \mathbin{\backslash\!\backslash} nd$ and $nd \mathbin{\backslash\!\backslash} b$.

Extended-real-valued functions and $D(X)$

Definition. Let X be a completely regular topological space. Then $D(X)$ denotes the set of continuous $\mathbb{R} \cup \{\pm\infty\}$-valued functions f on X such that $f^{-1}(\mathbb{R})$ is dense.

If $f, g \in D(X)$, then $f + g$ is defined and real-valued on $U := f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$. U is dense and open in X, but $f|_U + g|_U$ might not be the restriction to U of an element of $D(X)$. If there is $h \in D(X)$ such that $h|_U = f|_U + g|_U$, it is unique (since U is dense) and we say that $f + g$ is defined.

Suppose B is a subset of $D(X)$. If B contains 0 and is closed under $-$ and $\vee$, and for all $f, g \in B$, $f + g$ is defined, then B is an archimedean ℓ-group. In this case, we say *B is an ℓ-group of continuous extended-real-valued functions on X*, and we write $B \subseteq_\ell D(X)$.

The Yosida Theorem, Part 1

Theorem. Suppose A is an archimedean ℓ-group and $a \in A^+$. Then

(*i*) for all $b \in A$, $\widehat{b}_a \in D(Y(A, a))$,

(*ii*) $\widehat{A}_a := \{ \widehat{b}_a \mid b \in A \}$ is an ℓ-group of continuous extended-real-valued functions on $Y(A, a)$,

(*iii*) the map $\widehat{(\ )}_a : b \mapsto \widehat{b}_a : A \to \widehat{A}_a \subseteq D(Y(A, a))$ is an ℓ-homomorphism with kernel $a^\perp$,

(*iv*) $\widehat{a}_a =$ the constant function 1 on $Y(A, a)$.

Proof. (*i*) follows from the Theorem on slide 5. Ad (*ii*), if $b, c \in A$, then $\widehat{b}_a(M) + \widehat{c}_a(M) = \widehat{(b + c)}_a(M)$ for all M in a dense subset of $Y(A, a)$. Ad (*iii*), the map preserves the operations "on a dense subset of $Y(A, a)$;" that $a^\perp$ is the kernel follows from the Theorem on slide 5. (*iv*) follows from the definitions. □

The Yosida Theorem, Part 2: Functoriality

Theorem. Suppose $\phi : A \to A'$ is an ℓ-homomorphism, $a \in A$, and $a' = \phi(a) \in A'$. Let $Y(\phi) : Y(A', a') \to X(A)$(= the set of ℓ-prime ℓ-ideals of A) be defined by $Y(\phi)(M) := \phi^{-1}(M)$. Then:

(*i*) If $M \in Y(A', a')$, then $Y(\phi)(M) \in Y(A, a)$;

(*ii*) $Y(\phi)$ is continuous;

(*iii*) For all $g \in A$, $\widehat{\phi(g)}_{a'} = \widehat{g}_{a'} \circ Y(\phi)$.

Proof. Exercise.

The Yosida Theorem, Part 3: Idempotence

Theorem. Suppose X is a compact Hausdorff space and $A \subseteq_\ell D(X)$ is an ℓ-group of continuous extended-real-valued functions on X containing the constant function 1. Suppose further that for any two distinct points $x, y \in X$, there are $f, g \in A^+$ such that $f(x) = 0 = g(y)$ and $f(y) \neq 0 \neq g(x)$. For each $x \in X$, let $\mu(x) := \{ a \in A \mid a(x) = 0 \}$. Then $\mu : X \to Y(A, 1)$ is a homeomorphism, and $b \mapsto \widehat{b}_1$ is an ℓ-isomorphism of A with $\widehat{A}_1$.

The proof is a series of exercises:

1. For any two points distinct $x, y \in X$, there are $a, b \in A$ such that $x \in \operatorname{coz} a$, $y \in \operatorname{coz} b$ and $\cos a \cap \operatorname{coz} b = \emptyset$.
2. For each $x \in X$, $\{ \operatorname{coz} a \mid a \in A^+ \,\&\, a(x) \neq 0 \}$ is a neighborhood base at x.
3. μ is surjective.
4. μ is injective.
5. μ is a homeomorphism.
6. $\widehat{b}_1 = 0$ iff $b = 0$.

A research problem: Change of unit.

Suppose $a, b \in A$ and $0 \leq a \leq b$.

- What is the relationship between $Y(A, a)$ and $Y(A, b)$?
- Between the representation maps $\widehat{(\ )}_a$ and $\widehat{(\ )}_b$?
- What stronger statements are true when A is archimedean? When a and b are weak order units?