A Course on the Yosida Theorem

Classical & Pointfree Versions & Applications

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Lecture 10

Change of Unit

Examples

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Review

Localic Yosida. Suppose A is an archimedean ℓ-group, and $e \in A^+$.

(*i*) Let $y : A^+ \to \mathcal{Y}A$ be the set map universal for the following relations ($a, b \in A^+$):

- (I_1) $y(0) = \bot$,
- (I_2) $y(a \wedge b) = y(a) \wedge y(b)$,
- (I_3) $y(a + b) = y(a) \vee y(b)$,
- (I_4) $y(a \vee b) = y(a) \vee y(b)$.
- (Y) if $\{a_i\}_{i=0}^{\infty} \subseteq A^+$ and $a_i \uparrow_b a$, then $y(a) = \bigvee_{i=1}^{\infty} y(a_i)$.

Then $\mathcal{Y}A$ is order-isomorphic to the augmentation of the frame of archimedean kernels of A.

(*ii*) Let $y_e : A^+ \to \mathcal{Y}_e A$ be the quotient of $\mathcal{Y}A$ obtained by adding the relation:

- (U_e) $y_e(e) = \top$.

Then $\mathcal{Y}_e A$ regular Lindelöf.

(*iii*) Let $\Phi_e : A \to \mathcal{R}\,\mathcal{Y}_e A$ be defined by

$$\Phi_e(a)(p, q) = y_e\left((a - pe)^+ \wedge (qe - a)^+\right), \quad p, q \in \mathbb{Q}.$$

Then Φ_e is an ℓ-homomorphism with kernel $e^\perp$.

Much of this is constructively valid, but the constructive content of statements about the Lindelöf property and relatively-uniformly closed ideals is not known. See "Schlitt-1991.pdf", "Banaschewski-1999.pdf", "Banaschewski-Walters-Wayland-2007.pdf".

$\Phi_e(a) : \mathcal{R} \to \mathcal{Y}_e A$ is the "formal ratio of a to e." (Some observations)

(1) Suppose $A = C(X)$, with X a regular Lindelöf space. If $a, e \in C(X)$ and e is strictly positive on X, then $\Phi_e(a) : \mathcal{O}(\mathbb{R}) \to \mathcal{O}X$ is the map of open sets induced by the real-valued function a/e, i.e.

$$\Phi_e(a)(p, q) = \{\, x \in X \mid \tfrac{a(x)}{e(x)} \in (p, q) \,\}.$$

(2) If $y(f) = y(e)$, then $\mathcal{Y}_e A = \mathcal{Y}_f A$, and $\Phi_e(a)$ and $\Phi_f(e)$ both belong to $\mathcal{R}\,\mathcal{Y}_e A$. The proposition on the next slide will show that $\Phi_e(a) \cdot \Phi_f(e) = \Phi_f(a)$. (Formally, this resembles the equality $(a/e) \cdot (e/f) = a/f$.)

(3) If $y(f) \leqslant y(e)$, then there is a frame quotient map $\pi^e_f : \mathcal{Y}_e(A) \to \mathcal{Y}_f(A)$, and hence a "co-restriction" map $\pi^e_f \circ \_\_ : \mathcal{R}\mathcal{Y}_e A \to \mathcal{R}\mathcal{Y}_f A$:

$$\mathcal{R} \xrightarrow{\Phi_e(a)} \mathcal{Y}_e A \xrightarrow{\pi^e_f} \mathcal{Y}_f A \ .$$

Note that we may identify $\mathcal{Y}_e A$ (respectively, $\mathcal{Y}_f A$) with the interval $[\perp, y(e)]$ (respectively, $[\perp, y(f)]$) in $\mathcal{Y}A$, and $\pi^e_f(w) = w \wedge y(f)$ for any $w \in \mathcal{Y}_e A$.

Change of Unit. Proof, part 1.

Proposition (Change of Unit). If $a, e, f \in A^+$ and $y(f) \leqslant y(e)$, then $(\pi^e_f \circ \Phi_e(a)) \cdot \Phi_f(e) = \Phi_f(a)$.

Corollary. If $e, f \in A^+$ and $y(f) \leqslant y(e)$, then $(\pi^e_f \circ \Phi_e(f)) \cdot \Phi_f(e) = 1$.

Proof of Proposition. It suffices to show, for all $p, q > 0$:

$$(i) \quad \bigvee_{\substack{s \cdot u = p \\ s,u>0}} y_f \left[(a - se)^+ \wedge (e - uf)^+\right] = y_f\left((a - pf)^+\right), \text{ and}$$

$$(ii) \quad \bigvee_{\substack{t \cdot v = q \\ t,v>0}} y_f \left[(te - f)^+ \wedge (vf - e)^+\right] = y_f\left((qf - a)^+\right).$$

Note that for any $a, b \in A^+$ and any $k \in \mathbb{Q}_{>0}$, $y(a \wedge kb) = y(a) \wedge y(kb) = y(a) \wedge y(b) = y(a \wedge b)$. We use this to get line (2), below.

Ad (i), let $\mathrm{B} := (1/2)(a - pf)$ and let $\mathrm{A} := (1/2)(a + pf) = B + pf = -(B - a)$. Then:

$$\begin{aligned}
y_f\left[(a - se)^+ \wedge (e - uf)^+\right] &= y_f\left[(a - se)^+ \wedge \tfrac{1}{s}(se - pf)^+\right] && (1)\\
&= y_f\left[(((a - se) \wedge (se - pf)) \vee 0) - \mathrm{B} + \mathrm{B}\right] && (2)\\
&= y_f\left[(((\mathrm{A} - se) \wedge (se - \mathrm{A})) \vee -B) + \mathrm{B}\right] && (3)\\
&= y_f\left[(-|\mathrm{A} - se| \vee (-B)) + \mathrm{B}\right] && (4)\\
&= y_f\left[(\mathrm{B} - |\mathrm{A} - se|)^+\right] \leqslant y_f\left(B^+\right). && (5)
\end{aligned}$$

Thus, each $\bigvee$-term on the left of (i) is less the the right side of (i). *(Continued...)*

Change of Unit. Proof, part 2.

Or goal now is to show the inequality $\geqslant$ in the following:

$$\bigvee_{\substack{s\cdot u=p\\ s,u>0}} y_f\left[(a-se)^+ \wedge (e-uf)^+\right] = y_f\left((a-pf)^+\right). \tag{6}$$

To this end, define for $n, i \in \mathbb{N}$:

$$\color{red}{g(n,i)} := \left[(na-ie)\wedge(ie-npf)\right].$$

Note that $y_f\left(g(n,i)^+\right) = y_f\left[(a-\frac{i}{n}e)^+ \wedge (e-\frac{n}{i}pf)^+\right]$, since $y_f(a_1^+ \wedge a_2^+) = y_f\left((c_1a_1 \wedge c_2a_2)^+\right)$ for any positive rational numbers c_1, c_2. For the same reason, this is the $\bigvee$-term in the LHS of (6) with $s = i/n$.

$$\begin{aligned} g(n,i) &= \frac{na-npf}{2} + \left(\left(\frac{na+npf}{2}-ie\right)\wedge\left(ie-\frac{na+npf}{2}\right)\right) && (7)\\ &= \frac{na-npf}{2} - \left|\frac{na+npf}{2}-ie\right| && (8)\\ &= \frac{n}{2}\left((a-pf)-\left|(a+pf)-i\,\frac{2e}{n}\right|\right) && (9)\end{aligned}$$

Observe that (since $y_f(ca) = y_f(a)$):

$$y_f\left(g(n,i)^+\right) = y_f\left(\left((a-pf)-\left|(a+pf)-i\,\frac{2e}{n}\right|\right)^+\right).$$

Change of Unit. Proof concluded.

Let:

$$h := (a - pf) \tag{10}$$

$$h(n,i) := (a - pf) - \left|(a + pf) - i\,\frac{2e}{n}\right| \tag{11}$$

$$h(n) := \bigvee_{i=1}^{n^2}\left[(a - pf) - \left|(a + pf) - i\,\frac{2e}{n}\right|\right] \tag{12}$$

$$= (a - pf) - \bigwedge_{i=0}^{n^2}\left|(a + pf) - i\,\frac{2e}{n}\right| \tag{13}$$

Lemma. If $f, w, x \in A^+$, then: $\boxed{\bigwedge_{i=0}^{m} |f - i\,w| \leqslant (|f| - m\,w) \vee w}$, and $\boxed{(x - n\,w)^+ \wedge w \leqslant \frac{1}{n} x}$.

$$h - h(n) = \bigwedge_{i=0}^{n^2}\left|(a + pf) - i\,\frac{2e}{n}\right| \leqslant ((a + pf) - 2ne) \vee \frac{2e}{n} \tag{14}$$

$$(h - h(n))^+ \wedge e \leqslant \Big(((a + pf) - 2ne)^+ \wedge e\Big) \vee \frac{2e}{n} \leqslant \frac{(a + pf)}{2n} \vee \frac{2e}{n} \tag{15}$$

Lemma. If $a, b \in A$ and $e \in A^+$, $(a^+ \wedge e) - (b^+ \wedge e) \leqslant (a - b)^+ \wedge e$.

Using the lemma, we see that $h(n)^+ \wedge e$ converges to $h^+ \wedge e$ with regulator $(1/2)(a + pf) \vee 2e$. (We made a similar argument in Lecture 9.) This completes the proof of the inequality $\geqslant$ in (6). The proof of (ii) is similar. $\square$

Comments on the proof.

The proof of Change of Unit and of Preservation of Addition (in Lecture 9) use similar devices. In both cases, we prove, for a certain $g \in A$ and family $\{\, g_s \mid s \in \mathbb{Q} \,\}$, that

$$y_e(g^+) = \bigvee \{\, y_e(g_s^+) \mid s \in \mathbb{Q} \,\}. \tag{16}$$

We use the device of defining $h(n) := \bigvee \{\, h_{i/n} \mid i = -n^2, \ldots, n^2 \,\}$ (or $h(n) := \bigvee \{\, h_{i/n} \mid i = 1, \ldots, n^2 \,\}$), such that $y_e(g_s^+) = y_e(h_s^+)$ and observing that $\bigvee \{\, y_e(h_s^+) \mid s \in \mathbb{Q} \,\} = \bigvee_{n=1}^{\infty} h(n)$. Then, we show that $h(n) \to_{ru} h$.

The argument seems to be an elaboration of the simple proof of:

Lemma. For $a, e \in A^+$, $\bigvee_{n=1}^{\infty} y_e\left(\left(a - \frac{e}{n}\right)^+\right) = y_e(a)$.

Proof. $0 \leqslant a - \left(a - \frac{e}{n}\right)^+ \leqslant \frac{e}{n}$, so $\left(a - \frac{e}{n}\right)^+ \uparrow_e a$.

Challenge. Simplify the proofs of Preservation of Addition and Change of Unit, while preserving their constructive validity. Generalize to provide a (constructive) proof that if A is an f-ring with ring unit e, then $\Phi_e(a \cdot b) = \Phi_e(a) \cdot \Phi_e(b)$.

Elements of $\mathcal{RO}$ with multiplicative inverse.

[BH] "The inversion characterizations of $C(\mathcal{L})$ for a locale $\mathcal{L}$." *Rocky Mountain Journal of Mathematics* 49.7 (2019): 2107-2120. Library: "Ball-Hager-2019.pdf"

Suppose A is an archimedean ℓ-group with weak unit e.

Definition. [BH] An element $a \in A$ is said to be *kernel-maximal* if the **W**-kernel in A generated by a is all of A.

Our results shed light on the meaning of this:

Corollary to Change of Unit. Suppose A is an archimedean ℓ-group with weak unit e. The following are equivalent, for $f \in A^+$: (i) f is kernel-maximal, (ii) $y_e(f) = \top_{\mathcal{Y}_e(A)}$, (iii) $\mathcal{Y}_e(A) = \mathcal{Y}_f(A)$. If these are true, then $\Phi_e(f)$ has a multiplicative inverse in $\mathcal{RY}_e(A)$, namely $\Phi_f(e)$.

Much more is true:

Definition. [BH] An element $a \in A$ is said to be *Yosida-invertible* if there is some $f \in \Phi_e A$ such that $a \cdot f = 1$ (referring to the multiplication in $\mathcal{RY}_e A$).

Theorem. [BH] $\Phi_e A = \mathcal{RY}_e A$ iff A is divisible, uniformly complete, and every kernel-maximal element of A is Yosida-invertible.

Challenges. (1) Investigate uniform completion (Stone-Weierstrass) in the localic setting. (There is substantial lieterature on this.) (2) Conjecture: The group of invertible orthomorphisms of $\mathcal{RO}$ is $\{ f \in \mathcal{RO} \mid y(a) = y(1) \}$, and this is isomorphic to $\mathcal{RO}$ via the exponential map.