A Course on the Yosida Theorem

Classical & Pointfree Versions & Applications

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Lecture 1. Abelian ℓ-Groups Basics

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Abelian ℓ-groups: Definitions

Definition 1. An *abelian ℓ-group* is set A with operations 0, $-$, $+$, and $\vee$ where:

(*i*) $\langle A, 0, -, + \rangle$ is an abelian group;

(*ii*) $\langle A, \vee \rangle$ is join-semilattice (i.e., $\vee$ is associative, commutative and idempotent);

(*iii*) $\langle A, 0, -, +, \vee \rangle$ satisfies the distributive law

$$x + (y \vee z) = (x + y) \vee (x + z),$$

(i.e., each translation is a symmetry of $\langle A, \vee \rangle$).

Example. For any topological space X, $C(X)$, the set of all continuous real-valued functions on X, is an abelian ℓ-group under the pointwise operations. If L is a locale, $C(L)$ is an abelian ℓ-group.

Abelian ℓ-groups: Lattice structure

Definition 2. The operations $\wedge$, $()^+$, $()^-$, and $|\ |$ are defined by

- $x \wedge y := -\big((-x) \vee (-y)\big)$.
- $x^+ := x \vee 0$, $x^- := (-x) \vee 0$.
- $|x| := x \vee -x$.

Fact. $\langle A, \vee, \wedge \rangle$ is a distributive lattice; see the Exercises, below.

Remarks:

- By Definition 1, for every $b \in A$, the translation $a \mapsto b + a$ is a symmetry of $\langle A, \vee, \wedge \rangle$.
- By Definition 2, $\langle A, \vee, \wedge \rangle$ is "dually symmetric" under the reflection $a \mapsto -a$.

Exercises

1. Let $\langle L, \vee\rangle$ be a join semilattice. Define a relation $\leq$ on L by $a \leq b \iff: b = a \vee b$. Show that $\leq$ is a partial order, and for all $a, b \in L$, $a \vee b$ is the least upper bound of a and b.

In the remaining exercises A is an abelian ℓ-group and $\leq$ is the partial order on A induced by $\vee$, as in Exercise 1.

2. Show that for all $a, b, c, d \in A$: if $a \leq b$ and $c \leq d$ then $a + c \leq b + d$. (In particular, $a \leq b \iff 0 \leq b - a$.)

3. *Consequences of distributivity of $+$ over $\vee$.* Show that for all $a, b \in A$:

(*i*) $a = a^+ - a^-$.

(*ii*) $a^+ + b^+ \geq (a + b)^+$.

(*iii*) $(b - a)^+ \geq b^+ - a^+ \geq -((a - b)^+)$.

(*iv*) If $a \wedge b = 0$, then $(a - b)^+ = a$ and $(a - b)^- = b$.

(*v*) For all $n \in \mathbb{N}$, $n\,(a \vee 0) = n\,a \vee (n-1)\,a \vee \cdots \vee a \vee 0$.

(*vi*) From (*v*) deduce: if $0 < n \in \mathbb{N}$ and $0 \leq n\,a$ then $0 \leq a$.

Exercises (cont.)

4. *Properties of* $|\ |$. Show that for all $a, b \in A$:

(i) $0 \le |a|$. Hint. $a \le |a|$ and $-a \le |a|$. Use Exercises 2 and 3(vi).

(ii) $a^+ + a^- = |a|$.

(iii) $|a + b| \le |a| + |b|$.

5. Show that for all $x, y, z \in A$, $x + (y \wedge z) = (x + y) \wedge (x + z)$. Show that $a \mapsto -a$ is an order-reversing automorphism of A, and accordingly that $a \wedge b$ is the greatest lower bound of a and b; thus, $\langle A, \vee, \wedge \rangle$ is a lattice.

6. Suppose $a_i, b_j \in A$ and $a_i \wedge b_j = 0$ for $i = 1, \ldots m$ and $j = 1, \ldots n$. Show:

(i) $(a_1 + a_2) \wedge b_1 = 0$.

(ii) $(a_1 + \cdots + a_m) \wedge (b_1 + \cdots + b_n) = 0$.

7. Suppose $a, b \in A$ and $n \in \mathbb{N}$. Show:

(i) $(n\,a)^+ = n(a^+)$. Hint. $n\,a = n(a^+) - n(a^-)$. By 6(ii), $n(a^+) \wedge n(a^-) = 0$. Now use 3(iv).

(ii) $n\,(a \vee b) = n\,a \vee n\,b$, and $n\,(a \wedge b) = n\,a \wedge n\,b$.

Exercises (cont.)

8. Show that $\langle A, \vee, \wedge \rangle$ is a distributive lattice.
Hint. It is enough to show that $(x \wedge y)^+ = x^+ \wedge y^+$. The relation $(x \wedge y)^+ \leq x^+ \wedge y^+$ is immdiate. For the other inequality, let $z := (x \wedge y)^+ - (x \wedge y)$. Show $0 \leq x + z$ & $x \leq x + z$ and hence $x^+ \leq x + z$. Similarly $y^+ \leq y + z$. Thus $x^+ \wedge y^+ \leq (x \wedge y) + z = (x \wedge y)^+$.

9. Suppose $X \subseteq A$ is a set. Show that the set of all elements of A that can be written in the form

$$\bigvee_{i=1}^{p} \bigwedge_{j=1}^{q} \sum_{k=1}^{r} n_{ijk} x_{ijk},$$

where p, q, r, n_{ijk} are positive integers and $x_{ijk} \in X$, is closed under $-, +$ and $\vee$.

ℓ-ideals

Definition. Let A be an abelian ℓ-group. An *ℓ-ideal* of A is a subgroup $K \subseteq A$ such that $\vee$ induces a well-defined operation on A/K. Thus K is an ℓ-ideal if and only if

$$\forall a, a', b, b' \in A, \quad a - a', b - b' \in K \;\Rightarrow\; (a \vee b) - (a' \vee b') \in K.$$

Lemma 1. Let K be a subgroup of A. Then K is an ℓ-ideal if and only if:

$$\forall a, a' \in A, \quad a - a' \in K \;\Rightarrow\; a^+ - a'^+ \in K. \tag{1}$$

Proof. ($\Rightarrow$) is clear. ($\Leftarrow$) Suppose $a - a' \in K$. By (1) and the identity $a \vee b = (a - b)^+ + b$, we have $(a \vee b) - (a' \vee b) \in K$. If in addition, $b - b' \in K$, then $(a' \vee b) - (a' \vee b') \in K$, so $(a \vee b) - (a' \vee b') \in K$.

ℓ-ideals

Theorem. Suppose A is an abelian ℓ-group and K is a subgroup of A. Then K is an ℓ-ideal if and only if:

(i) $\forall\, x, y \in K,\ x \vee y \in K$ (i.e., K is *sup-closed*), and

(ii) $\forall\, x, y \in K, z \in A$, if $x \leq z \leq y$ then $z \in K$ (i.e., K is *convex*).

A sub-ℓ-group of A is a subgroup of A that is sup-closed (not necessarily convex).

Proof. ($\Rightarrow$) Assume K is an ℓ-ideal. If $x, y \in K$, then

$$(x \vee y) + K = (x + K) \vee (y + K) = (0 + K) \vee (0 + K) = 0_{A/K},$$

so $x \vee y \in K$. Thus, K is sup-closed. If $x, y \in A$, and $x \leq y$, then $x \vee y = y$, so $(x + K) \vee (y + K) = y + K$, so $x + K \leq y + K$, i.e., $a \mapsto a + K$ is order-preserving. Accordingly, if $x, y \in K$, $z \in A$, and $x \leq z \leq y$, then $0 \leq x + K \leq z + K \leq y + K = 0_{A/K}$, so $z + K = 0_{A/K}$, so $z \in K$. Thus, K is convex.

ℓ-ideals

Proof of theorem (concluded). ($\Leftarrow$) Suppose K is a subgroup of A that is sup-closed and convex. By Lemma 1, to show that K is an ℓ-ideal, it suffices to show that $x - y \in K$ implies that $x^+ - y^+ \in K$. So, suppose $x - y \in K$. Then also $y - x \in K$, and by the sup-closed assumption, $(x - y)^+ \in K$ and $(y - x)^+ \in K$. By convexity and Exercise 6(*iii*), $x^+ - y^+ \in K$. $\square$

The lattice of ℓ-ideals

It is clear that any intersection of ℓ-ideals of A is an ℓ-ideal of A. Thus, every subset X of A is contained in a smallest ℓ-ideal, namely, the intersection of all ℓ-ideals containing X. This is denoted $\langle X \rangle$. We write $\langle y \rangle$ for the smallest ℓ-ideal containing y and $\langle X, y \rangle$ for the smallest ℓ-ideal containing $X \cup \{y\}$.

Proposition. If $K \subseteq A$ is an ℓ-ideal and $a \in A$, then

$$\langle K, a \rangle = \{\, x \in A \mid \exists k \in K,\, \exists n \in \mathbb{N} :\; 0 \leq |x| \leq k + n\,|a| \,\}.$$

Proof. The containment $\supseteq$ is evident. Therefore, it suffices to show that the right hand side is an ℓ-ideal, i.e., that it is a subgroup of A, that it is closed under $\vee$ and that it is convex. The details are left as an exercise.

The lattice of ℓ-ideals

For ℓ-ideals $J, K \subseteq A$, we define $J \wedge K := J \cap K$ and $J \vee K := \langle J \cup K \rangle$.

Proposition. Suppose $a, b, c \in A^+$.

1. $\langle a \wedge b \rangle = \langle a \rangle \wedge \langle b \rangle$.
2. $\langle a \vee b \rangle = \langle a + b \rangle = \langle a \rangle \vee \langle b \rangle$.
3. $\langle c \rangle \wedge \big(\langle a \rangle \vee \langle b \rangle \big) = \big(\langle c \rangle \wedge \langle a \rangle \big) \vee \big(\langle c \rangle \wedge \langle b \rangle \big)$.

Proof. Exercise.

Remark. This shows that the principal ℓ-ideals of A form a distributive lattice, which we denote dA. It has bottom $\langle 0 \rangle$, but does not have a top element. The frame of all ℓ-ideals of A is isomorphic to the frame of all lattice ideals of dA.

ℓ-prime ℓ-ideals

Def. $A^+ := \{ a \in A \mid 0 \leq a \}$

Fact. The following are equivalent:

1. A is totally ordered;
2. for all $a, b \in A$, $a \wedge b \in \{a, b\}$;
3. for all $a, b \in A$, $a \wedge b = 0$ implies $a = 0$ or $b = 0$;
4. for all $a, b \in A^+$, $a \wedge b = 0$ implies $a = 0$ or $b = 0$.

Fact. Suppose $K \subseteq A$ is an ℓ-ideal. Then A/K is totally ordered if and only if: for all $a, b \in A^+$, $a \wedge b \in K$ implies $a \in K$ or $b \in K$.

Definition. We say that an ℓ-ideal K is *ℓ-prime* if satisfies the equivalent conditions of the previous fact.

Maximal ℓ-ideals

Theorem. Suppose $a \in A$ and K is an ℓ-ideal maximal among those not containing a. Then K is ℓ-prime.

Proof. Suppose $x, y \in A^+ \setminus K$. Then $|a| \leq k + n\,x$ and $|a| \leq k + n\,y$ for some $0 \leq k \in K$ and $n \in \mathbb{N}$. (Note that we can find k and n that work for both x and y.) Thus, $|a| - k \leq n\,x \wedge n\,y = n(x \wedge y)$. Therefore $a \in \langle K, x \wedge y \rangle$, which implies that $x \wedge y \in A^+ \setminus K$. □

What's next?

In the next lecture, we will look at archimedean ℓ-groups. Here are the important topics:

- Maximal ℓ-ideals in an archimedean ℓ-group
- The space of e-maximal ℓ-ideals
- The cover of an e-maximal ℓ-ideal
- Hölder's Theorem
- The Yosida Representation Theorem