Intersection Cohomology

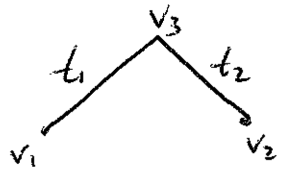
Lecture notes

March 17, 2009

Charlie Egedy

Two strategies for defining the sheaf $\underline{M}$ on the inverted vee.

A.



M_{v_1} generated by g_1 in degree -1

M_{v_2} generated by P

M_{v_3} generated by $g_1 + Qt_1$ subject to $g_1 + Qt_1 \equiv P \mod t_2$

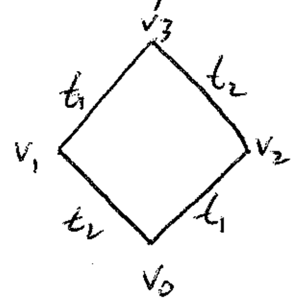
Note that if $g_1 \equiv P \mod t_1$, that is $P = g_1 + g_2 t_1$, then we have

$-g_2 t_1 + Qt_1 \equiv 0 \mod t_2$, so $Q - g_2 \equiv 0 \mod t_2$ and $Q = g_2 + t_2 g_3$

$\therefore$ M_{v_3} is generated by $g_1 + g_2 t_1 + g_3 t_1 t_2$

(If P in degree -1 also, then g_2 in degree -2).

B. Add a phantom vertex (ie, complete the crystallographic graph)



M_{v_0} generated by g_0 in degree -2

M_{v_1} generated by $g_0 + t_2 g_1$

M_{v_2} generated by $g_0 + t_1 g_2$

M_{v_3} generated by $g_0 + t_1 g_2 + Qt_2$ subject to

$$g_0 + t_1 g_2 + Qt_2 \equiv g_0 + t_2 g_1 \mod t_1 \implies$$

$$t_2(Q - g_1) \equiv 0 \mod t_1 \implies$$

$$Q = g_1 + t_1 g_3$$

So M_{v_3} generated by $g_0 + t_1 g_2 + t_2 g_1 + t_1 t_2 g_3$

Now delete v_0 to obtain

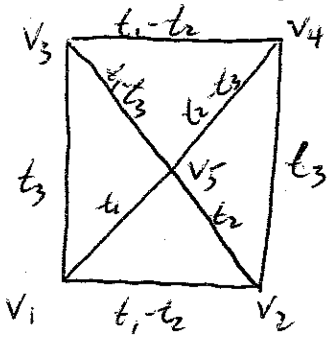
$$\left.\begin{array}{l} M_{v_1} \text{ generated by } t_2 g_1 \\ M_{v_2} \text{ generated by } t_1 g_2 \end{array}\right\} \text{in the same degree}$$

M_{v_3} generated by $t_1 g_2 + t_2 g_1 + t_1 t_2 g_3$

(13)

The Egyptian Pyramid

A. The following projection assists in assigning orientations to edges:



(Choose a basis for $\mathbb{R}^3$ and let it define the other orientations)

M_{V_1} generated by g_1

M_{V_2} generated by $g_1+(t_1-t_2)g_2$

M_{V_3} generated by g_1+t_3P

M_{V_4} generated by $g_1+t_3P+(t_1-t_2)g_2+(t_1-t_2)t_3g_4'$ (applying part B above)

M_{V_5} generated by $g_1+t_3P+(t_1-t_2)g_2+(t_1-t_2)t_3g_4'+Q(t_2-t_3)$ subject to

(a) $(t_1-t_2)g_2+(t_1-t_2)t_3g_4+Q(t_2-t_3)\equiv 0 \mod (t_1-t_3) \Rightarrow$

$(t_1-t_2)g_2+(t_1-t_2)t_1g_4+Q(t_2-t_1)\equiv 0 \mod (t_1-t_3)\Rightarrow$

$(t_1-t_2)[g_2+t_1g_4'-Q]\equiv 0 \mod (t_1-t_3) \Rightarrow$

$$Q=g_2+t_1g_4'+(t_1-t_3)Q'$$

Thus, we have M_{V_5} generated by (after substitution & simplification)

$$g_1+t_3P+(t_1-t_3)g_2+t_2(t_1-t_3)g_4'+Q'(t_1-t_3)(t_2-t_3)$$

(14)

(b) $t_3 P + (t_2 - t_3) g_2 + t_2(t_1 - t_3) g_4' + Q'(t_1 - t_3)(t_2 - t_3) \equiv 0 \mod t_2 \Rightarrow$

$t_3 P - t_3 g_2 - t_3(t_1 - t_3) Q' \equiv 0 \mod t_2$

We need $P - g_2 \equiv 0 \mod t_1 - t_3$, so set $P = g_2 + P'(t_1 - t_3)$

$\Rightarrow t_3(t_1 - t_3)[P' - Q'] \equiv 0 \mod t_2 \Rightarrow Q' = P' + Q'' t_2$

Thus, after simplification, M_{v_5} is generated by

$g_1 + t_1 g_2 + P'(t_1 - t_3) t_2 + t_2(t_1 - t_3) g_4' + Q'' t_2 (t_1 - t_3)(t_2 - t_3)$

(c) $t_1 g_2 + P'(t_1 - t_3) t_2 + t_2(t_1 - t_3) g_4' + Q'' t_2 (t_1 - t_3)(t_2 - t_3) \equiv 0 \mod t_1$

$\Rightarrow -t_2 t_3 P' - t_2 t_3 g_4' + t_2(-t_3)(t_2 - t_3) Q'' \equiv 0 \mod t_1$

$\Rightarrow P' + g_4' \equiv 0 \mod t_2 - t_3$. So set $P' = g_3$ and

$g_4' = g_4(t_2 - t_3) - g_3$

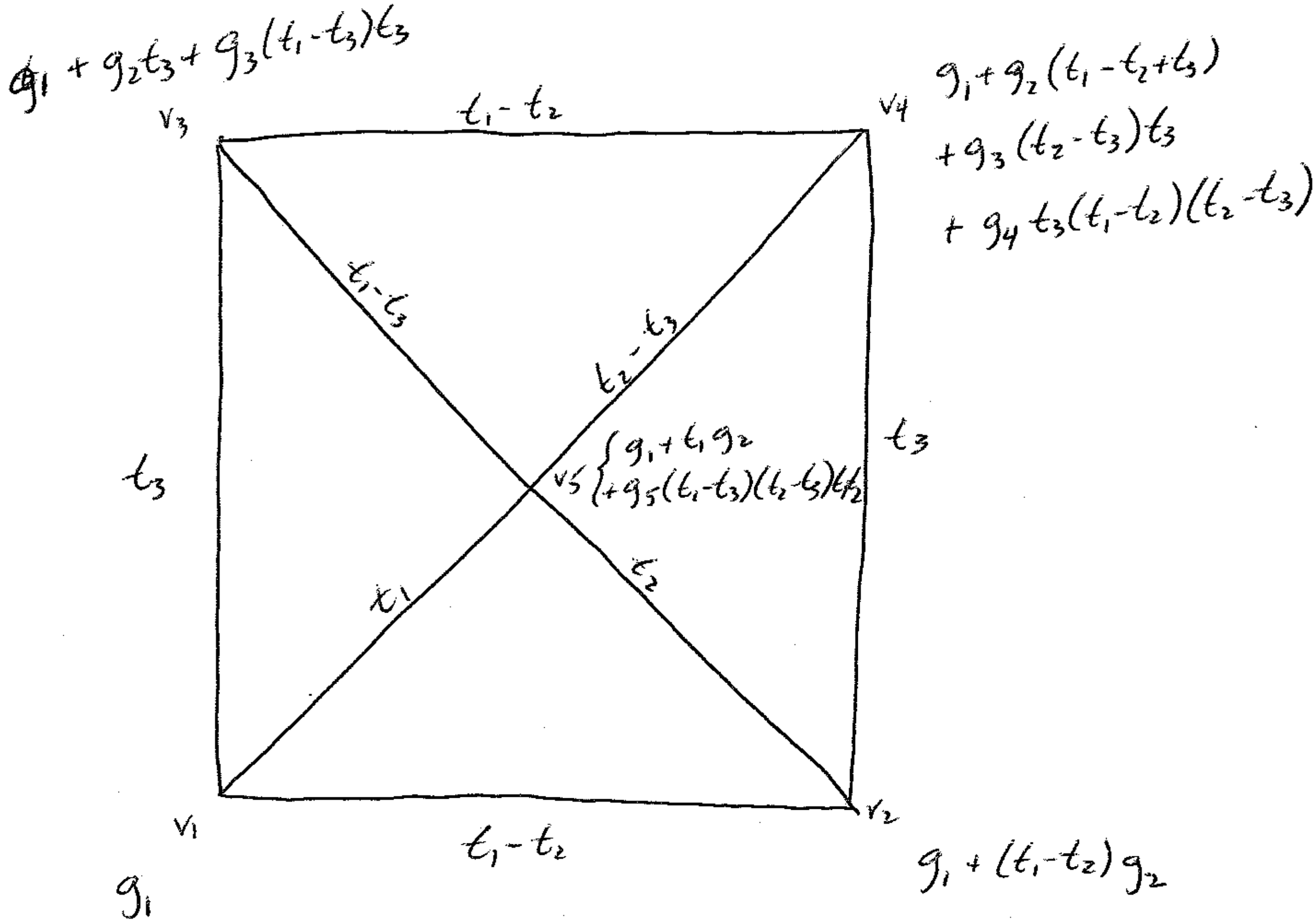
$\Rightarrow -t_2 t_3 (t_2 - t_3)[g_4 + Q''] \overset{\equiv 0 \mod t_1}{\Rightarrow} Q'' = -g_4 + g_5 t_1$

So M_{v_5} is generated by:

$t_1 g_2 + (t_1 - t_3) t_2 g_3 + t_2(t_1 - t_3)[g_4(t_1 - t_3) - g_3] + (g_5 t_1 - g_4) t_2 (t_1 - t_3)(t_2 - t_3)$

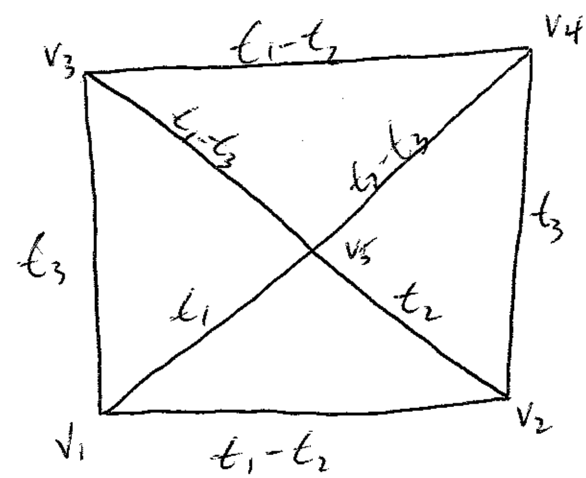
$= g_1 + t_1 g_2 + g_5 (t_1 - t_3)(t_2 - t_3) t_1 t_2$

We summarize in the following diagram:


$g_1 + g_2 t_3 + g_3(t_1 - t_3)t_3$
v_3
$t_1 - t_2$
v_4
$g_1 + g_2(t_1 - t_2 + t_3)$
$+ g_3(t_2 - t_3)t_3$
$+ g_4 t_3(t_1 - t_2)(t_2 - t_3)$
$t_1 - t_3$
$t_2 - t_3$
t_3
v_5
$g_1 + t_1 g_2$
$+ g_5(t_1 - t_3)(t_2 - t_3)t_1 t_2$
t_3
t_1
t_2
v_1
$t_1 - t_2$
v_2
g_1
$g_1 + (t_1 - t_2) g_2$

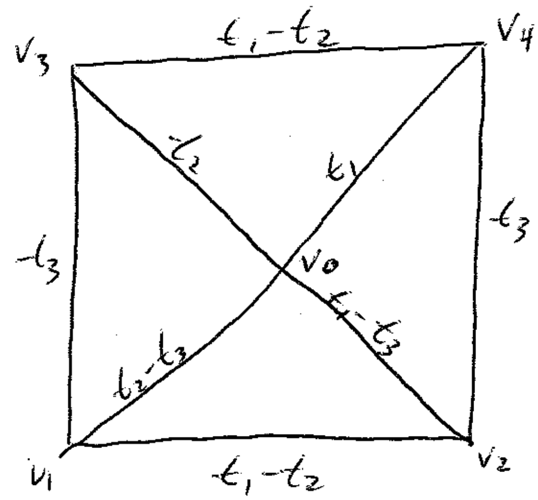
B. We now build a new sheaf on the Egyptian pyramid by adding a vertex v_0. The two following projections will be useful:

upper pyramid



(Same as in A.)

lower (inverted) pyramid



M_{v_0} generated by g_0

M_{v_1} generated by $g_0 + (t_2-t_3)g_1$

M_{v_2} generated by $g_0 + (t_2-t_3)g_1 + Q(t_1-t_2)$

subject to $(t_2-t_3)g_1 + Q(t_1-t_2) \equiv 0 \mod (t_1-t_3)$

$\Rightarrow (t_2-t_1)g_1 + Q(t_1-t_2) \equiv 0 \mod (t_1-t_3)$

$\Rightarrow Q = g_1 + g_2(t_1-t_3)$

$\therefore M_{v_2}$ generated by $g_0 + (t_1-t_3)g_1 + (t_1-t_2)(t_1-t_3)g_2$

Similarly, M_{v_3} generated by $g_0 + t_2 g_1 + t_2 t_3 g_3$

(17)

M_{v_4} generated by $g_0 + t_2 g_1 + t_2 t_3 g_3 + (t_1 - t_2) Q$ subject to

(a) $(t_2 + t_3 - t_1) g_1 + t_2 t_3 g_3 - (t_1 - t_2)(t_1 - t_3) g_3 + Q(t_1 - t_2) \equiv 0 \mod t_3 \Rightarrow$

$(t_2 - t_1) g_1 - (t_1 - t_2) t_1 g_2 + Q(t_1 - t_2) \equiv 0 \mod t_3 \Rightarrow$

$(t_1 - t_2)[Q - g_1 - t_1 g_2] \equiv 0 \mod t_3 \Rightarrow$

$Q = g_1 + t_1 g_2 + Q' t_3 \Rightarrow$

(b) $t_2 g_1 + t_2 t_3 g_3 + (t_1 - t_2)[g_1 + t_1 g_2 + Q' t_3] \equiv 0 \mod t_1$

$\Rightarrow t_2 g_1 + t_2 t_3 g_3 - t_2 g_1 - t_2 t_3 Q' \equiv 0 \mod t_1$

$\Rightarrow t_2 t_3 [g_3 - Q'] \equiv 0 \mod t_1$

$\Rightarrow Q' = g_3 + g_4 t_1$

$\Rightarrow Q = g_1 + t_1 g_2 + t_1 t_3 g_4$

$\Rightarrow M_{v_4}$ is generated by

$$g_0 + t_1 g_1 + t_1 (t_1 - t_2) g_2 + t_1 t_3 g_3 + t_1 t_3 (t_1 - t_2) g_4$$

We no longer need v_0, so:

M_{v_1} generated by $(t_2 - t_3) g_1$

M_{v_2} generated by $(t_1 - t_3) g_1 + (t_1 - t_2)(t_1 - t_3) g_2$

M_{v_3} generated by $t_2 g_1 + t_2 t_3 g_3$

M_{v_4} generated by $t_1 g_1 + t_1 (t_1 - t_2) g_2 + t_1 t_3 g_3 + t_1 t_3 (t_1 - t_2) g_4$

(18)

M_{v_5}: generated by $t_1 g_1 + t_1(t_1-t_2)g_2 + t_1 t_3 g_3 + t_1 t_3 (t_1-t_2) g_4 + Q(t_2-t_3)$

subject to:

(a) $g_1(t_1-t_2+t_3) + t_1(t_1-t_2)g_2 + t_1 t_3 g_3 + t_1 t_3(t_1-t_2)g_4 + Q(t_2-t_3) \equiv 0 \mod t_1$

$\Rightarrow (t_3-t_2)g_1 + Q(t_2-t_3) \equiv 0 \mod t_1 \Rightarrow Q = g_1 + Q' t_1$

(b) $t_3 g_1 + t_1 t_3 g_2 + t_1 t_3 g_3 + t_1^2 t_3 g_4 + (g_1 + Q' t_1)(-t_3) \equiv 0 \mod t_2$

$\Rightarrow t_1 t_3 [g_2 + g_3 + t_1 g_4 - Q'] \equiv 0 \mod t_2$

$\Rightarrow Q' = g_2 + g_3 + t_1 g_4 + Q'' t_2$

(c) $(t_1-t_2)g_1 + t_1(t_1-t_2)g_2 + (t_1-t_2)t_3 g_3 + t_1 t_3(t_1-t_2)g_4$

$+ (t_2-t_3)[g_1 + t_1 g_2 + t_1 g_3 + t_1^2 g_4 + t_1 t_2 Q''] \equiv 0 \mod t_1 - t_3$

$\Rightarrow (t_1-t_3)[g_1 + t_1 g_2 + t_2 g_3 + t_1 t_2 (t_1-t_3) g_4] + (t_2-t_3) t_1 t_2 Q'' \equiv 0 \mod t_1 - t_3$

$\Rightarrow Q'' = g_5(t_1-t_3)$

Thus, M_{v_5} is generated by

$g_1(t_1-t_3) + g_2(t_1-t_3)t_1 + g_3 t_2 (t_1-t_3) + g_4(t_1-t_3)t_1 t_2 + t_1 t_2 (t_1-t_3)(t_2-t_3) g_5$