Finding Patterns In The Powers Of a Change Of Basis Matrix

SMILE @ Louisiana State University

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Finding Patterns In The Powers Of a Change Of Basis Matrix

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Outline

Definitions

Stirling Numbers of the 2^{nd} Kind, $S(n,k)$

The number of ways to partition $[n]$ into k blocks, with $S(0,0) = 1$.

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Example

[3], a set with three elements, $\{a, b, c\}$. There are three ways to partition this set into two blocks. We take $\{a\}$ and $\{b, c\}$, $\{a, b\}$ and $\{c\}$, or $\{a, c\}$ and $\{b\}$. Thus $S(3, 2) = 3$.

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The number of partitions of $[n]$, or

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Example

Using our previous example, $S(3,1) = 1$, $S(3,2) = 3$, and $S(3,3) = 1$. So $B(3) = 5$.

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Complementary Bell Numbers, $\tilde{B}(n)$

The number of partitions with an even number of blocks minus the number of partitions with an odd number of blocks, or

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Example

$$\tilde{B}(3) = S(3,2) - (S(3,1) + S(3,3)) = 3 - (1+1) = 1.$$

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It has been proven that there is at most 1 exception.

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where $\lambda_j(x)$ are polynomials of degree j defined recursively by

- $\lambda_0(k) = 1$
- $\lambda_{j+1} = x\lambda_j(x) - \lambda(x+1)$

The Polynomials $\lambda_j(x)$

- $\tilde{B}(n+j) = \sum_{k=0}^{n} (-1)^k \lambda_j(k) S(n,k) \implies \lambda_j(0) = \tilde{B}(j)$.

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- There is an infinite matrix P such that $c_{j+1} = Pc_j$
- $P^n(0,0) = \tilde{B}(n)$.

Defining P

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- $P(r, r+1) = -r-1$
- $P(r, r) = r-1$
- $P(r+1, r) = 1$
- $P(r, s) = 0$ if $|r-s| > 1$

$$P = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & \dots \\ 1 & 0 & -2 & 0 & 0 & \dots \\ 0 & 1 & 1 & -3 & 0 & \dots \\ 0 & 0 & 1 & 2 & -4 & \dots \\ 0 & 0 & 0 & 1 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} \text{aw+by} & \text{ax+bz} \\ \text{cw+ dy} & \text{cx+dz} \end{pmatrix}$$

Matrix Multiplication

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- Then $AB(r,s) = \sum_{k=1}^{m} A(r,k)B(k,s)$.
- $P^2(r,s) = \sum_{k=0}^{\infty} P(r,k)P(k,s)$

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$$= P(r,r-1)P(r-1,r) + P(r,r)P(r,r) + P(r,r+1)P(r+1,r)$$

$$= 1(-(r-1)-1) + (r-1)(r-1) + (-r-1)1$$

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$$= r^2 - 4r$$

The matrix P^2 has the following form

- $P^2(r, r+2) = (r+1)(r+2)$
- $P^2(r, r+1) = -2r^2 - r + 1$
- $P^2(r, r) = r^2 - 4r$
- $P^2(r+1, r) = 2r - 1$
- $P^2(r+2, r) = 1$
- $P^2(r, s) = 0$ if $|r - s| > 2$

$$P^2 =$$

$$\begin{bmatrix} r^2-4r & -2r^2-r+1 & (r+1)(r+2) & 0 & \dots \\ -1 & r^2-4r & -2r^2-r+1 & (r+1)(r+2) & \dots \\ 1 & 2r-1 & r^2-4r & -2r^2-r+1 & \dots \\ 0 & 1 & 2r-1 & r^2-4r & \dots \\ 0 & 0 & 1 & 2r-1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$P^2 =$

$$\begin{bmatrix} 0 & 1 & 2 & 0 & ... \\ -1 & -3 & -2 & 6 & ... \\ 1 & 1 & -4 & 9 & ... \\ 0 & 1 & 3 & -3 & ... \\ 0 & 0 & 1 & 5 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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- $P^3(r, r+3) = -(r+1)(r+2)(r+3)$
- $P^3(r, r+2) = 3r^3 + 9r^2 + 6r$
- $P^3(r, r+1) = -3r^3 + 3r^2 + 8r + 2$
- $P^3(r, r) = r^3 - 9r^2 + 6r + 1$
- $P^3(r+1, r) = 3r^2 - 6r - 2$
- $P^3(r+2, r) = 3r$
- $P^3(r+3, r) = 1$ for all $(r - s) = 3$
- $P^3(r, s) = 0$ if $|r - s| > 3$

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- $P^4(r, r+4) = (r+1)(r+2)(r+3)(r+4)$
- $P^4(r, r+3) = -4r^4 - 26r^3 - 56r^2 - 46r - 12$
- $P^4(r, r+2) = 6r^4 + 14r^3 - 5r^2 - 23r - 10$
- $P^4(r, r+1) = -4r^4 + 14r^3 + 20r^2 + r - 1$
- $P^4(r, r) = r^4 - 16r^3 + 30r^2 + -4r + 1$
- $P^4(r+1, r) = 4r^3 - 18r^2 - 2r + 1$
- $P^4(r+2, r) = 6r^2 - 4r - 5$
- $P^4(r+3, r) = 4r + 2$
- $P^4(r+4, r) = 1$ for all $(r - s) = 4$
- $P^4(r, s) = 0$ if $|r - s| > 4$

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Theorem

$P^j(r,s) = 0$ *if* $|r-s| > j$ *for all* $j \in \mathbb{N}$

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Theorem

$P^j(r,r+j) = (-1)^j(r+1)^{(j)}$ for all $j \in \mathbb{N}$, where $(r+1)^{(j)}$ denotes the rising factorial, $(r+1)(r+2)\ldots(r+j)$.

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- Assume that $P^j(r, r+j) = (-1)^j (r+1)^{(j)}$ for some $j \in \mathbb{N}$.

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- Assume that $P^j(r, r+j) = (-1)^j (r+1)^{(j)}$ for some $j \in \mathbb{N}$.
- Then

$$\begin{aligned} P^{j+1}(r, r+j+1) &= \sum_{k=0}^{\infty} P^j(r,k) P(r+j, r+j+1) \\ &= P^j(r, r+j) P(r+j, r+j+1) \\ &= (-1)^j (r+1)^{(j)} (-r-j-1) \\ &= (-1)^{j+1} (r+1)^{(j)} (r+j+1) \\ &= (-1)^{j+1} (r+1)^{(j+1)} \end{aligned}$$

□

Some experimenting

$$P_5^{j=2} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & -3 & -2 \\ 1 & 1 & -4 \end{pmatrix}$$

$$P_5^3 = \begin{pmatrix} 1 & 2 & 0 & -6 \\ -2 & -1 & 10 & 18 \\ 0 & -5 & -15 & 6 \\ 1 & 3 & -2 & -35 \end{pmatrix}$$

$$P_5^4 = \begin{pmatrix} 1 & -1 & -10 & -12 & 24 \\ 1 & 12 & 30 & -18 & -144 \\ -5 & -15 & 1 & 129 & 132 \\ 2 & -3 & -43 & -92 & 236 \\ 1 & 6 & 11 & -59 & -303 \end{pmatrix}$$

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Some experimenting

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$\frac{1}{-1} = -1$ $\frac{2}{1} = 2$

$\frac{P_m^{j=2}(0,r+j)}{P_m^{j=2}(r+j,0)} = \frac{2}{1} = 2$

$$P_5^3 = \begin{pmatrix} 1 & 2 & 0 & -6 \\ -2 & -1 & 10 & 18 \\ 0 & -5 & -15 & 6 \\ 1 & 3 & -2 & -35 \end{pmatrix}$$

$\frac{2}{-2} = -1$ $\frac{0}{0} = dne$ $\frac{-6}{1} = -6$

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$$P_5^4 = \begin{pmatrix} 1 & -1 & -10 & -12 & 24 \\ 1 & 12 & 30 & -18 & -144 \\ -5 & -15 & 1 & 129 & 132 \\ 2 & -3 & -43 & -92 & 236 \\ 1 & 6 & 11 & -59 & -303 \end{pmatrix}$$

$\frac{-1}{1} = -1$ $\frac{-10}{-5} = 2$ $\frac{-12}{2} = -6$

$\frac{24}{1} = 24$

$\frac{P_m^{j=4}(0,r+j)}{P_m^{j=4}(r+j,0)} = \frac{24}{1} = 24$

Does this look familiar?

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$$-1,\ 2,\ -6,\ 24,\ -120,\ 720,\ \ldots = -1^r(r!)$$

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or the Alternating Factorials

How to use this formula for our matrix

$$-1^r(r!)P^j_m(r,0) = P^j_m(0,r), r > 0$$

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General case for all rows and columns

$$-1^{r-s}(r!)P^j_m(r,s) = s!P^j_m(r,s), r \geq 0$$

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  - Consider P^j modulo $3 * 2^k$ for different values of k.

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