SAMPLING THEORY

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Abstract. Sampling theory is the study of the reconstruction of a function from its values (samples) on some subset of the domain of the function. In general, the setting is a vector space V of functions over some domain $\mathbb{X}$ for which it is possible to evaluate functions. In this article, we shall explore properties within sampling theory from the point of view of linear transforms and inner products.

1. Background

Sampling theory involves the analysis of unique vector spaces that exhibit the reconstruction property. In discussing sampling theory, we first begin with reviewing some simple background. Recall the term vector space. A real vector space is a set, V, satisfying the following properties:

(1) **Commutativity:** $u + v = v + u$ where both $u, v \in V$;

(2) **Associativity:** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ where $u, v, w \in V$ and a and b are scalars;

(3) **Additive Identity:** There exist an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;

(4) **Additive Inverse:** For every $v \in V$, there is a $w \in V$ such that $v + w = 0$;

(5) **Multiplicative Identity:** $1v = v$ for all $v \in V$;

(6) **Distributive Properties:** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathrm{F}$ (scalars) and all $u, v \in V$

Specific sets of vectors utilized in signal reconstruction are known as ***frames***. A frame is constructed and identified using an inequality of upper and lower frame bounds as well as inner product spaces.

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Definition 1.1. Given a vector space, V, an ***inner product*** on V is a function over the field $\mathbb{F}$, creating linear map $(\cdot|\cdot) : V \times V \to \mathbb{F}$. Let $u, v, w \in V$ and $\lambda \in \mathbb{F}$, an inner product will satisfy the following properties:

(1) **Conjugate Symmetry:** $(u|v) = \overline{(v|u)}$

(2) **Linearity:**
$$(\lambda u|v) = \lambda(u|v)$$
$$\text{and}$$
$$(u+v|w) = (u|w) + (v|w)$$

(3) **Positive:** $(u|u) \geq 0$ and $(u|u) = 0$ only for $u = 0$

For spanning sets, the following lemma may be useful:

Lemma 1.2. *Let V be a finite dimensional vector space of dimension n. If $\{v_i\}_{i=1}^n$ is a set vectors that span V and, for all i, $v_i \in V$, then $\{v_i\}_{i=1}^n$ is a basis.*

Proof. Suppose not, meaning $\{v_i\}_{i=1}^n$ is not a basis. Then $\{v_i\}_{i=1}^n$ spans V, but is not linearly independent. That is, there exists at least one vector v_k in $\{v_i\}_{i=1}^n$ such that v_k can be written as a linear combination of the other vectors in $\{v_i\}_{i=1}^n$. If we remove such vectors from $\{v_i\}_{i=1}^n$, we are left with a set, W, of linearly independent vectors that span V. Thus W is a basis of V with less than n vectors. This is a contradiction since a basis must have the same number of vectors as the dimension of its vector space. Therefore $\{v_i\}_{i=1}^n$ is linearly independent and thus a basis. $\square$

2. Framework

Definition 2.1. If we let $\{v_1, \ldots, v_k\}$ be any set of vectors, it is then classified as a frame if there are numbers $A, B > 0$ such that for all $v \in V$ the following inequality holds

$$A\|v\|^2 \leq \sum_{i=1}^{k} \mid (v, v_i) \mid^2 \leq B\|v\|^2.$$

A frame in V can then be classified as just a spanning set of that particular vector space.

Proposition 2.2. *The set $\{v_1, \ldots, v_k\}$ is said to be a frame if and only if it is a spanning set of V.*

Proof. We prove by contradiction that $\{v_i\}_{i=1}^{k}$ spans$V$. Assume $\{v_,1\ldots,v_k\}$ is not a spanning set of V. Then there would be $v \in V$ such that $(v|v_i) = 0$ for all $1 \leq i \leq k$. Meaning

$$
\begin{aligned}
A\|v\|^2 &\leq \sum_{i=1}^{k} |(v|v_i)|^2 = 0 \\
&= \sum_{i=1}^{k} 0 \\
&= 0
\end{aligned}
$$

This creates a contradiction because $A = B = 0$ but $A, B > 0$ in the definition of a frame. Therefore all frames are spanning sets.
From the opposite direction if $\{v_1, \ldots, v_k\}$ spans$V$ we can show that the given set is a frame.There is always a $B > 0$, such that $\sum |(v|v_i)|^2 \leq B\|v\|^2$ exist. For the following statements,

$$
\begin{aligned}
\sum |(v|v_i)|^2 &\leq \sum \|v\|^2 \cdot \|v_i\|^2 \\
&= \left(\sum_{i=1}^{k} \|v_i\|^2\right) \|v\|^2 \\
&= B\|v\|^2
\end{aligned}
$$

can be claimed as valid since from the Cauchy-Schwarts inequality this particular relation follows through as true . Therefore, there always exist an upper frame bound, B. Now, suppose $v \neq 0$ then for all $A > 0$, there is a $v \in V$ such that

$$
\begin{aligned}
A &\leq \sum_{i=1}^{k} \left| \left(\frac{v}{\|v\|} | v_i \right) \right|^2 \\
A &\leq \sum_{i=1}^{k} \frac{|(v|v_i)|^2}{\|v\|^2} \\
A\|v\|^2 &\leq \sum_{i=1}^{k} |(v|v_i)|^2.
\end{aligned}
$$

Which shows an existence of A as the lower frame bound and concludes that $\{v_1 \ldots v_k\}$ is a frame. $\square$

There are two types of frames, a ***tight frame*** and a ***parseval frame***. If $A = B$ the frame $\{v_1, \ldots, v_k\}$ is a tight frame, but if $A = B = 1$ then it is said to be a parseval frame. For every frame there exists what is known as a ***dual frame***, as seen in the following theorem.

Theorem 2.3. *Suppose $\{v_i\}_{i=1}^k$ is a frame, then their exists $\{w_i\}_{i=1}^k \subset V$ such that for all $v \in V$*

$$v = \sum_{i=1}^{k} (v|v_i)w_i = \sum_{i=1}^{k} (v|w_i)v_i.$$

Where the set $\{w_i\}_{i=1}^k$ is called a dual frame of $\{v_i\}_{i=1}^k$.

With every dual frame there also exists what is known as the ***canonical dual frame***. In order to define the canonical dual frame we first have to review the properties of both a set of sampling and uniqueness.

Definition 2.4. If $\{v_i\}_{i=1}^k \subset V$ it is considered ***unique*** if there is only one object fulfilling its properties. Particularly indicating the set is determined by a certain set of data.

The canonical dual frame not only incorporates a property of uniqueness but involves the function $\Theta_{\mathbb{X}}$ as well. $\Theta_{\mathbb{X}}$ is defined as the linear map of $V \to \mathbb{C}^k$, where vector inner products produce a complex number set known as the analysis operator

$$\Theta(v) = \begin{pmatrix} (v|v_1) \\ (v|v_2) \\ \vdots \\ (v|v_k) \end{pmatrix}.$$

Note, the analysis operator $\Theta_{\mathbb{X}}$ is in fact a linear transformation. The adjoint, Θ^*, creates the adjoint map, reproducing the vectors. This leads into frame operator S which the canonical frame uses in its connection and relation to other dual frames. S is then said to be

$$S = \Theta^*\Theta$$

which means

$$\begin{aligned} S^* &= \Theta^*\Theta^{**} \\ S^* &= \Theta^*\Theta \\ S &= S^* \end{aligned}$$

and

$$Sv = \Theta^*\Theta v$$

Therefore operator S is found to be equal to its adjoint and is invertible. This makes S adjoint and its inverse, S^{-1}, self-adjoint, allowing the canonical dual frame to be written in terms of S^{-1}. The canonical dual frame exist when $w_i = S^{-1}v_i$. A key property of all dual frames that ties into this particular property is that they are unique and injective.

Proposition 2.5. *If we define $\mathbb{P}_N$ to be*

$$\mathbb{P}_N = \left\{ f : \mathbb{R} \to \mathbb{R} : f(x) = \sum_{n=0}^{N} a_n x^n, a_n \in \mathbb{R} \right\}$$

which is a set of polynomials of degree N or less, then set $\mathbb{X}$ is a set of uniqueness for $\mathbb{P}_N$ if and only if $\Theta_{\mathbb{X}}$ is injective.

Proof. If we let set $\{x_1 \dots x_n\} = \mathbb{X}$, we then look at two different functions restricted to only these particular vectors, denoted as $f|_{\mathbb{X}}$ and $g|_{\mathbb{X}}$. Assume set $\mathbb{X}$ is unique. Then $f, g \in \mathbb{P}_N$, claiming each function as elements of $\mathbb{P}_N$, and $f|_{\mathbb{X}} = g|_{\mathbb{X}}$ by definition of uniqueness. We can then say $\Theta_{\mathbb{X}}(f) = \Theta_{\mathbb{X}}(g)$ and the calculated values from each function is equal to one another. Therefore the following is true:

$$\Theta_{\mathbb{X}}(f) = \Theta_{\mathbb{X}}(g)$$

$$\begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix} = \begin{pmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{pmatrix}$$

$$f(x_n) = g(x_n)$$

Thus both functions f and g are indeed equal to one another and $\Theta_{\mathbb{X}}$ is one-to-one.Proven in the opposite direction. Supposing that $\Theta_{\mathbb{X}}$ is injective. Since $\Theta_{\mathbb{X}}$ is unique that means that $\Theta_{\mathbb{X}}(f) = \Theta_{\mathbb{X}}(g)$ since the x_i's ,where $0 \leq i \leq N$, only produce one calculated value. We can then conclude $f = g$ which means $f|_{\mathbb{X}} = g|_{\mathbb{X}}$ and $\mathbb{X}$ is a set of uniqueness. □

As a consequence of Proposition 2.6, the next two theorems come into play.

Theorem 2.6. *Let $\mathbb{X} = \{x_0, x_1, \dots, x_N\} \subset \mathbb{R}$; then $\mathbb{X}$ is a set of uniqueness for $\mathbb{P}_N$.*

Proof. From Proposition 2.1, it suffices to show that $\Theta_{\mathbb{X}}$ is one-to-one. We can express the matrix form of $\Theta_{\mathbb{X}}$, denoted $[\Theta_{\mathbb{X}}]$, using $\{x^i\}_{i=0}^{N}$ as the basis for $\mathbb{P}_N$ and the standard basis for $\mathbb{R}^{N+1}$ denoted as $\{e_i\}_{i=0}^N$. We know that the ij^{th} element in $[\Theta_{\mathbb{X}}]$ is the j^{th} basis vector of $\mathbb{P}_N$ evaluated at the i^{th} element of $\mathbb{X}$. So

$$[\Theta_{\mathbb{X}}] = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ 1 & x_1 & x_1^2 & \dots & x_1^N \\ 1 & x_2 & x_2^2 & \dots & x_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{bmatrix}$$

For $[\Theta_{\mathbb{X}}]$ to be one-to-one, the above $(N+1)\times(N+1)$ matrix must be invertible and thus the determinant must be nonzero. Since $[\Theta_{\mathbb{X}}]$ is the Vandermonde matrix and can be shown

$$\mathbf{det}([\boldsymbol{\Theta}_{\mathbb{X}}]) = \prod_{\mathbf{0 \leq i \leq j \leq N}} (\mathbf{x_j} - \mathbf{x_i}),$$

the determinant is known to always be nonzero so the transform is one-to-one and $\mathbb{X}$ is a set of uniqueness. □

By the definiton of S, the matrix representation of S can be derived using the matrix previously defined. Thus

$$[S] = \begin{bmatrix} N+1 & \sum_{i=0}^N x_i & \sum_{i=0}^N x_i^2 & \dots & \sum_{i=0}^N x_i^N \\ \sum_{i=0}^N x_i & \sum_{i=0}^N x_i^2 & \sum_{i=0}^N x_i^3 & \dots & \sum_{i=0}^N x_i^{N+1} \\ \sum_{i=0}^N x_i^2 & \sum_{i=0}^N x_i^3 & \sum_{i=0}^N x_i^4 & \dots & \sum_{i=0}^N x_i^{N+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^N x_i^N & \sum_{i=0}^N x_i^{N+1} & \sum_{i=0}^N x_i^{N+2} & \dots & \sum_{i=0}^N x_i^{2N} \end{bmatrix}$$

With math based computer software (i.e. Mathematica), it is possible to generate S^{-1} for a particular set $\mathbb{X}$ and then use that S^{-1} to generate the canonical dual frame of any frame that spans $\mathbb{P}_N$.

Example:

Consider $\mathbb{P}_2$ and $\mathbb{X} = \{-2, 0, 1\}$. Let our frame be denoted as $\{(3,0,0), (0,1,0), (0,0,2)\}$ where for a polynomial (a, b, c) corresponds to $a+bx+cx^2$. The following Mathematica commands provide an outline.

```
r1 = -2;

r2 = 0;

r3 = 1;


theta = {{1, r1, r1^2}, {1, r2, r2^2}, {1, r3, r3^2}}
```

$\{\{1,-2,4\},\{1,0,0\},\{1,1,1\}\}$

%16//MatrixForm

$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

adjoint = Transpose$[\{\{1,r_1,r_1\text{^}2\},\{1,r_2,r_2\text{^}2\},\{1,r_3,r_3\text{^}2\}\}]$

$\{\{1,1,1\},\{-2,0,1\},\{4,0,1\}\}$

%17//MatrixForm

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 4 & 0 & 1 \end{pmatrix}$$

S **= adjoint.theta**

$\{\{3,-1,5\},\{-1,5,-7\},\{5,-7,17\}\}$

%22//MatrixForm

$$\begin{pmatrix} 3 & -1 & 5 \\ -1 & 5 & -7 \\ 5 & -7 & 17 \end{pmatrix}$$

InS = Inverse$[S]$

$\left\{\left\{1,-\frac{1}{2},-\frac{1}{2}\right\},\left\{-\frac{1}{2},\frac{13}{18},\frac{4}{9}\right\},\left\{-\frac{1}{2},\frac{4}{9},\frac{7}{18}\right\}\right\}$

%29//MatrixForm

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{13}{18} & \frac{4}{9} \\ -\frac{1}{2} & \frac{4}{9} & \frac{7}{18} \end{pmatrix}$$

InS.$\{3, 0, 0\}$

$\left\{3, -\frac{3}{2}, -\frac{3}{2}\right\}$

InS.$\{0, 2, 0\}$

$\left\{-1, \frac{13}{9}, \frac{8}{9}\right\}$

InS.$\{0, 0, 1\}$

$\left\{-\frac{1}{2}, \frac{4}{9}, \frac{7}{18}\right\}$

Theorem 2.7. *Let $\mathbb{X} = \{x_0, x_1, \ldots, x_N\} \subset \mathbb{R}$; then $\mathbb{X}$ is a set of sampling for $\mathbb{P}_N$.*

Proof. By Theorem 2.6, assuming for some $f \in \mathbb{P}_N$ for which we know the values of f at points in $\mathbb{X}$, since $\mathbb{X}$ is a set of uniqueness we must provide a reconstruction algorithm to find f. One such algorithm is known as Lagrange interpolation, which generates the following polynomials:

$$p_{x_j}(x) = \prod_{k=0,\, k\neq j}^{N} \frac{(x - x_k)}{(x_j - x_k)}$$

where each polynomial has degree N and satisfies the conditions $p_{x_j}(x_j) = 1$ and $p_{x_j}(x_l) = 0$ for $l, j = 0, 1, \ldots, N-1$ where $l \neq j$. Thus $g(x) = \sum_{j=0}^{N} f(x_j)p_{x_j}(x)$ is an element $\mathbb{P}_N$ and $f(x_j) = g(x_j)$ for all j. Since $\mathbb{X}$ is a set of uniqueness, $f(x) = g(x)$ and therefore

$$f(x) = \sum_{j=0}^{N} f(x_j)p_{x_j}(x).$$

$\square$

Note: It should be said that an alternative method of reconstructing f would be to apply the inverse of $\Theta_{\mathbb{X}}$ to a column vector that consist of values of f at points in $\mathbb{X}$. From this point forward, the Lagrange polynomials will be denoted as $\{p_j(x)\}_{j=0}^N$ where $p_j = p_{x_j}$.

Claim: The Lagrange polynomials $\{p_j(x)\}_{j=0}^N$ form a basis.

Proof. It suffices to show $\{p_j(x)\}_{j=0}^N$ spans $\mathbb{P}_N$. Consider an arbitrary polynomial $p(x)$ in $\mathbb{P}_N$. Then $p(x) = \sum_{j=0}^N c_j\, x^j$. If we allow $c_j = d_j \sum_{k=0}^N p_{jk}$, then

$$\begin{aligned} p(x) &= \sum_{j=0}^N x^j\, d_j \sum_{k=0}^N p_{jk} \\ &= \sum_{j=0}^N d_j \sum_{k=0}^N p_{jk}\, x^j \\ &= \sum_{j=0}^N d_j \sum_{k=0}^N p_{jk}\, x^k \\ &= \sum_{j=0}^N d_j\, p_j(x) \end{aligned}$$

. So $p(x)$ can be written as a linear combination of Lagrange polynomials, so $\{p_j(x)\}_{j=0}^N$ spans $\mathbb{P}_N$ and, by Lemma 2.2, is a basis. □

Theorem 2.8. *Gram-Schmidt Theorem: Let $\mathcal{V}$ be a finite dimensional vector space, and let $\{x_i\}_{i=1}^k$ be a linearly independent set in $\mathcal{V}$. Then there exists an orthonormal set $\{u_i\}_{i=1}^k$ with the same span.*

Proof. Let the set of vectors $\{v_i\}_{i=1}^k$ be defined recursively.
Let $v_j = x_j - \sum_{i=1}^{j-1} \frac{(x_j|v_i)}{\|v\|^2}\, v_i$ for $j = 1,\, 2,\, \ldots,\, k$.
This definition gives $\{v_1\} = \{x_1\}$ with each vector $\{v_j\}$ in the span of $\{x_i\}_{i=1}^j$. Due to the linear independence of the original collection of vecors, any vector $\{v_j\} \neq 0$. Thus it can be verified that $(v_i \mid v_j) = 0$ for $i \neq j$. Therefore, $\{v_j\}_{j=1}^k$ is an orthogonal set. Define $\{u_i\}$ as $\{u_i\} = \frac{v_i}{\|v_i\|}$. Then $\{u_i\}_{i=1}^k$ is an orthonormal set with the same span as $\{x_i\}_{i=1}^k$. □

With the use of the frame operator and this specific dual frame property, the unique canonical dual frame can be used to express any set frame. Having the ability to express dual frames in terms of themselves, we can further use the *Riesz Representation Theorem* to connect reconstructive vectors used in sampling.

Theorem 2.9. *Riesz Representation Theorem: If φ is an inner product space of the finite dimensional vector space, V, where φ is a linear map, $\varphi : V \to \mathbb{F}$, and $\varphi \,\epsilon\, V^*$, where V^* is the dual space of V, then there exist $v_\varphi \,\epsilon\, V$ for all $v \,\epsilon\, V$ such that*

$$\varphi(v) = (v|v_\varphi)$$

Proof. Let $\{e_i\}_{i=1}^n$ be any orthonormal basis for $\mathcal{V}$. Define $v_\varphi = \sum_{i=1}^n \overline{\varphi(e_i)}\, e_i$. Let $w \in V$, $w = \sum_{i=1}^n c_i\, e_i$, and $\varphi(w) = \sum_{i=1}^n c_i \varphi(e_i)$. Note that $c_i = (w \mid e_i)$. The matrix for $\varphi : \mathcal{V} \to \mathbb{F}$ with respect to the basis $\{e_i\}_{i=1}^n$ is the $1 \times n$ matrix $[\varphi e_1\ \varphi e_2\ \ldots\ \varphi e_n]$. Taking the inner product of w and v_φ gives

$$\begin{aligned}
(w \mid v_\varphi) &= \left(\sum_{i=1}^n c_i\, e_i \mid \sum_{j=1}^n \overline{\varphi(e_j)}\, e_j \right) \\
&= \sum_{i,j=1}^n \left(c_i\, e_i \mid \overline{\varphi(e_j)}\, e_j \right) \\
&= \sum_{i,j=1}^n c_i\, \varphi(e_j)\, (e_i \mid e_j) \\
&= \sum_{i=1}^n c_i\, \varphi(e_i) \quad \text{(since } \{e_1\}_{i=1}^n \text{ is an orthonormal basis)} \\
&= \varphi\left(\sum_{i=1}^n c_i\, e_i \right) \\
&= \varphi(w).
\end{aligned}$$

□

3. Discoveries

Assuming we use the standard dot product as our inner product, we shall explore the certain properties. As previously mentioned, $\Theta_{\mathbb{X}}(f)$ evaluates the function f at each of the points in $\mathbb{X}$. Thus by the Riesz Representation Theorem,

$$\begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix} = \begin{pmatrix} (f|\, f_{x_0}) \\ (f|\, f_{x_1}) \\ \vdots \\ (f|\, f_{x_N}) \end{pmatrix}.$$

Note that for $f, f_{x_j} \in \mathbb{P}_N$, $f(x) = \sum_{j=0}^{N} a_j x^j$ and $f_{x_j}(x) = \sum_{j=0}^{N} b_j x^j$. So for x_i in $\mathbb{X}$,

$$f(x_i) = a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_N x_i^N$$

Using the inner product previously defined,

$$\begin{aligned} (f|\, f_{x_i}) &= \sum_{j=0}^{N} a_j\, b_j \\ &= a_0\, b_0 + a_1 b_1 + a_2 b_2 + \ldots + \ a_N b_N \end{aligned}$$

Therefore for each component,

$$f(x_i) = (f|\, f_{x_i})$$

which implies

$$a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_N x_i^N = a_0\, b_0 + a_1 b_1 + a_2 b_2 + \ldots + \ a_N b_N$$

Suppose we choose a term and its complement on the other side of the equation, say $a_0 = a_0\, b_0$. Dividing both sides by a_0, we get $1 = b_0$. For $a_1 x_i = a_1 b_1$, we have $x_i = b_1$. We continue this process such that $b_j = x_i^j$. Therefore we can write each f_{x_j} as

$$f_{x_i}(z) = \sum_{j=0}^{N} x_i^j\, z^j$$

Claim: $\{f_{x_i}\}_{i=0}^{N}$ is a frame

Proof. It suffices to show that the set spans $\mathbb{P}_N$. Recall that any polynomial in $\mathbb{P}_N$ has the form $p(x) = \sum_{j=0}^{N} a_j x^j$, where we take $\{x^j\}_{j=0}^{N}$ to be the standard basis for $\mathbb{P}_N$ and $a_j \in \mathbb{R}$. Allowing $a_j = c_j\, \sum_{r=0}^{N} x_j^r$

$$\begin{aligned} p(x) &= \sum_{j=0}^{N} x^j c_j\, \sum_{r=0}^{N} x_j^r \\ &= \sum_{j=0}^{N} c_j\, \sum_{r=0}^{N} x_j^r\, x^j \\ &= \sum_{j=0}^{N} c_j\, \sum_{r=0}^{N} x_j^r\, x^r \\ &= \sum_{j=0}^{N} c_j\, f_{x_j}(x) \end{aligned}$$

Thus we can write any polynomial in $\mathbb{P}_N$ as a linear combination of of vectors in $\{f_{x_i}\}_{i=0}^N$, so $\{f_{x_i}\}_{i=0}^N$ spans $\mathbb{P}_N$ and is a frame. $\square$

Since the Lagrange polynomials were shown to be a basis, Theorem 3.4 can be used to confirm if $\{f_{x_i}\}_{i=0}^N$ is a dual frame and thus the canonical dual frame by the following proposition.

Proposition 3.1. *Let $\{x_i\}_{i=0}^N$ be a basis for a finite-dimensional inner product space. Then its dual frame is unique.*

Let $f(x)$ exist in $\mathbb{P}_N$ such that $f(x) = \sum_{j=0}^N f(x_j)\, p_j(x)$, where $x_j \in \mathbb{X}$ and $p_j(x)$ be the j^{th} Lagrange polynomial written in the standard basis of $\mathbb{P}_N$ defined as $p_j(x) = \sum_{k=0}^N p_{jk}\, x^k$. By the Riesz Representation Theorem, $f(x) = \sum_{j=0}^N (f|\, f_{x_j})\, p_j(x)$. Thus

$$\begin{aligned}
\sum_{j=0}^N (f|\, f_{x_j})\, p_j(x) &= \sum_{j=0}^N \sum_{r=0}^N x_r^j\, f(x_r) \sum_{k=0}^N p_{rk}\, p_j(x) \\
&= \sum_{j=0}^N \sum_{r=0}^N x_r^j\, f(x_r) \sum_{k=0}^N p_{rk} \sum_{s=0}^N p_{js}\, x^s \\
&= \sum_{j=0}^N \sum_{r=0}^N \sum_{k=0}^N \sum_{s=0}^N x_r^j\, f(x_r)\, p_{rk}\, p_{js}\, x^s
\end{aligned}$$

Whereas

$$\begin{aligned}
\sum_{j=0}^N (f|\, p_j)\, f_{x_j}(x) &= \sum_{j=0}^N \sum_{r=0}^N p_{jr}\, f(x_r) \sum_{k=0}^N p_{rk}\, f_{x_j}(x) \\
&= \sum_{j=0}^N \sum_{r=0}^N p_{jr}\, f(x_r) \sum_{k=0}^N p_{rk} \sum_{s=0}^N x_j^s \\
&= \sum_{j=0}^N \sum_{r=0}^N \sum_{k=0}^N \sum_{s=0}^N x_r^j\, f(x_r)\, p_{rk}\, p_{js}\, x^s
\end{aligned}$$

Since $f(x) = \sum_{j=0}^N (f|\, f_{x_j})\, p_j(x) = \sum_{j=0}^N (f|\, p_j)\, f_{x_j}(x)$ for all f, $\{f_{x_i}\}_{i=0}^N$ is not only a dual frame for the Lagrange polynomials, by Proposition 2.1, it is the only dual frame and therefore is it's canonical dual frame.

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