Instructions: *Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems.* Start each problem on a new page, and write the problem number on the **top right** corner of the page. Make sure you order the pages correctly before submitting the exam. You have three hours. Good luck!

A. **Point Set Topology** (2 problems)

A1. Let X and Y be topological spaces, and let $f : X \to Y$ be a function. Prove that the following two statements are equivalent.

i. For every open subset V of Y, the set $f^{-1}(V)$ is open in X.

ii. For all $A \subseteq X$, the following inclusion holds: $f(\overline{A}) \subseteq \overline{f(A)}$.

What property of f do these equivalent statements define?

A2. Prove that a compact Hausdorff space is normal. (Recall that a space X is normal if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets of X containing A and B respectively.)

A3. Consider the set of rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$.

i. Find the components and path components of $\mathbb{Q}$. Is $\mathbb{Q}$ connected? path connected?

ii. Show that $\mathbb{Q}$ is not locally connected and not locally path connected.

iii. Do any of your answers to the above questions change if you instead use the discrete topology on $\mathbb{Q}$?

B. **Algebraic Topology** (2 problems)

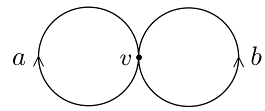
B1. Compute the fundamental group of S^2 with n points removed, where n is a positive integer. Also compute the fundamental group of $\mathbb{R}^3$ with the three coordinate axes removed. Justify your computations and recall any results you are using.

B2. Let $f : X \to Y$ be continuous. Let x_1 and x_2 be points of X, and let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. There are induced homomorphisms $f_{1*} : \pi_1(X, x_1) \to \pi_1(Y, y_1)$ and $f_{2*} : \pi_1(X, x_2) \to \pi_1(Y, y_2)$. Show that if X is path-connected then there are isomorphisms $\phi : \pi_1(X, x_1) \to \pi_1(X, x_2)$ and $\psi : \pi_1(Y, y_1) \to \pi_1(Y, y_2)$ such that the following diagram commutes.

$$\begin{array}{ccc} \pi_1(X, x_1) & \xrightarrow{f_{1*}} & \pi_1(Y, y_1) \\ \downarrow \phi & & \downarrow \psi \\ \pi_1(X, x_2) & \xrightarrow{f_{2*}} & \pi_1(Y, y_2) \end{array}$$

Make sure to prove that ϕ and ψ are isomorphisms. You may assume the standard properties of the path concatenation operation $*$.

B3. Let X denote the space below, and identify $\pi_1(X, v)$ with the free group $F(a, b)$.



i. Exhibit two 3-sheeted covers of X: a regular (normal) cover X_N and a irregular (non-normal) cover X_I. Explain how you know X_N and X_I have the desired properties.

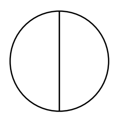
ii. For each of X_N and X_I, describe the complete list of deck transformations.

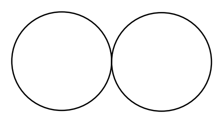
iii. Let $p : X_I \to X$ denote the covering map (where X_I is the irregular cover you constructed). Choose $w \in p^{-1}(v)$. Find a minimal set of generators for the (free) subgroup $p_*(\pi_1(X_I, w))$ of $F(a, b)$.

C. **Mixed** (1 problem)

C1. i. Define what it means for two spaces to have the same *homotopy type.*

ii. Do the following two spaces have the same homotopy type? Explain.





iii. If X and Y have the same homotopy type and X is compact must Y also be compact? Sketch a proof or give a counterexample.

iv. If X and Y have the same homotopy type and X is path connected must Y also be path connected? Sketch a proof or give a counterexample.

C2. The suspension of a topological space X is the quotient space $\Sigma X = X \times [0, 1]/\sim$ where $(x, t) \sim (y, s)$ if and only if either $(x, t) = (y, s)$ or $s = t = 1$ or $s = t = 0$.

i. Prove that the suspension ΣX need not be simply connected.

ii. Prove that the suspension ΣX is simply connected if X is path connected. (Hint: Use the Seifert–van Kampen Theorem.)