Topology **Comprehensive/Qualifying Exam** **August 2024**

Instructions: *Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems.* Be sure to write the number of each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

A. **Point Set Topology** (2 problems)

A1. Let X and Y be topological spaces, and let $\pi_1 : X \times Y \to X$ be the first projection. Prove or disprove each of the following statements.

i. The projection π_1 is a continuous map.

ii. The projection π_1 is an open map.

iii. The projection π_1 is a closed map.

A2. Let X and Y be topological spaces and $f : X \to Y$ a surjective continuous function.

i. Prove that if X is compact, then Y is compact.

ii. Prove that if X is connected, then Y is connected.

A3. i. Show that if $f\colon X \to Y$ is a continuous bijection from a compact space X to a Hausdorff space Y, then f is a homeomorphism.

ii. Exhibit an example of topological spaces X and Y and a continuous bijection $f\colon X \to Y$ that is *not* a homeomorphism.

B. **Homotopy** (2 problems)

B1. Let X be a contractible topological space.

i. Show that X is path-connected.

ii. Show that if A is a retract of X, then A is also contractible.

B2. i. Show that every continuous map $f : \mathbb{R}P^2 \to S^1$ is nullhomotopic.

ii. Find a continuous map $f : S^1 \times S^1 \to S^1$ that is not nullhomotopic.

B3. i. Use the Seifert-van Kampen Theorem to show that the n-sphere is simply connected for $n \geq 2$.

ii. Show that $\mathbb{R}^{n+1} \setminus \{0\}$ is simply connected when $n \geq 2$.

C. **Mixed** (1 problem)

C1. Prove that no two of the following three spaces are homeomorphic:

i. the unit sphere S^2

ii. the plane $\mathbb{R}^2$

iii. the space $\mathbb{R}^3$

C2. Let $X = S^1 \times I$ be the cylinder, and let $Y = \mathbb{R}^2 - \{(0,0)\}$ be the punctured plane.

i. Is X homotopy equivalent to Y? Justify your answer.

ii. Is X homeomorphic to Y? Justify your answer.
