Introduction to Applied Mathematics
Qualifying Exam, January 2025

The exam is 3 hours. Each problem is worth 20 points, and you can do a maximum of five problems for a total potential score of 100 points. Pick at least one from each category.

$C_c^\infty(\Omega)$ is defined as smooth compactly supported functions with the closure of the support in Ω.

Category I: Continuum Mechanics

1. **(20 points)** We denote $x = \varphi(X,t) = \varphi_t(X)$ the transformation from material to spatial coordinates. For a field $\Phi(x,t)$ in spatial coordinates, we denote

$$\frac{d}{dt}\Phi(x,t) = \frac{\partial}{\partial t}[\Phi(\varphi(X,t),t)]\Big|_{X=\varphi_t^{-1}(x)}.$$

Consider a body B. Let $S(x,t)$ be the Cauchy stress tensor, $v(x,t)$ the velocity spatial field, and $\rho(x,t)$ the mass density spatial field, and $\rho(x,t)b(x,t)$ the body force field. Assume all fields smooth. Recall the balance equations in spatial coordinates

$$\rho\frac{d}{dt}v = \nabla\cdot S + \rho b \tag{1}$$

$$S = S^T. \tag{2}$$

Prove that for any $\Omega \subset B$ open with smooth boundary, and $\Omega_t := \varphi_t(\Omega)$, then

$$\int_{\Omega_t} \rho v\cdot\frac{d}{dt}v\, dV_x + \frac{1}{2}\int_{\Omega_t} S:(\nabla^x v + \nabla^x v^T)\, dV_x = \int_{\partial\Omega_t} v\cdot Sn\, dA_x + \int_{\Omega_t}\rho b\cdot v\, dV_x.$$

Here $A:B$ is understood as the inner product between second order tensors A and B. Recall $(\nabla\cdot S)_i = \sum_j \frac{\partial}{\partial x_j}S_{ij}$.

2. We denote $x = \varphi(X,t) = \varphi_t(X)$ the transformation from material to spatial coordinates. For a vector field $\Phi(x,t)$ in spatial coordinates, we denote $\frac{d}{dt}\Phi(x,t) = \frac{\partial}{\partial t}[\Phi(\varphi(X,t),t)]\Big|_{X=\varphi_t^{-1}(x)}$.

Let $\Phi(x,t)$ be a vector field in spatial coordinates.

- **a.** **(10 points)** Show $\frac{d}{dt}\Phi = \frac{\partial}{\partial t}\Phi + (\nabla^x\Phi)\Phi$.
- **b.** **(10 points)** Let $\Omega\subset B$ be open, and $\Omega_t = \varphi_t(\Omega)$. Find $G(x,t)$ such that

$$\frac{d}{dt}\int_{\Omega_t}\Phi(x,t)\, dx = \int_{\Omega_t} G(x,t)\, dx.$$

3. (20 points) Let the vector fields E and H in $\mathbb{R}^3$ (with coordinates (x_1, x_2, x_3)) satisfy the free harmonic Maxwell system

$$\nabla \times E = i\omega\mu H \tag{3}$$

$$\nabla \times H = -i\omega\epsilon E \tag{4}$$

Suppose that E and H are independent of x_3, that H is perpendicular to the (x_1, x_2)-plane, and that ϵ and μ are smooth scalar functions of x_1 and x_2 alone, and $\epsilon(x_1, x_2) \neq 0$. Show that the Maxwell system can be reduced to a single scalar second-order PDE for H.

Category II: Fourier Analysis

4. (20 points)
Find the vector-valued field $u : \mathbb{R}^3 \to \mathbb{R}^3$ solving the PDE

$$-\triangle u + \nabla(\nabla \cdot u) + u = f$$

where the components of f are in Schwartz space. Prove the components of u are also in Schwartz space.

5.

a. (10 points) Suppose $f \in C^{(k)}_{\text{per}}([-1/2, 1/2])$, i.e. k-times continuously differentiable and periodic. Recall

$$\hat{f}(n) = \langle f, \phi_n \rangle, \qquad \phi_n(x) = e^{2\pi i x n}.$$

Show for $n \neq 0$ that there exists a constant $C > 0$ such that

$$|\hat{f}(n)| \leq C|n|^{-k}.$$

b. (10 points) Prove if $k > 1$ that the partial Fourier sum $S_N(f) \to f$ pointwise uniformly on $x \in [-1/2, 1/2]$.

6. (20 points) Suppose $g \in C(\mathbb{R})$ be periodic with period 1, i.e. $g(x+1) = g(x)\ \forall x \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ be irrational. Define the sequence

$$G_N = \frac{1}{2N+1} \sum_{n=-N}^{N} g(\alpha n).$$

Prove

$$G_N \to \int_{-1/2}^{1/2} g(x)dx \qquad \text{as } N \to \infty.$$

Category III: Weak-form PDEs & Distribution Theory

7. (20 points) Let $\phi \in C_c^\infty(\mathbb{R}^d)$. Let $\phi_\epsilon(x) = \varepsilon^{-d}\phi(x/\epsilon)$, and assume $\int \phi(x)\, dx = 1$. Find the limit of ϕ_ϵ as $\epsilon \to 0$ in the sense of distributions.

8. (20 points) Let $\Omega \subset \mathbb{R}^3$ be a simply connected open set with smooth boundary. Let 1_Ω be the distribution satifying $\langle 1_\Omega, \phi\rangle = \int_\Omega \phi(x)\, dx$. Find the distributional gradient $\nabla 1_\Omega$ over vector-valued test functions.

9. (20 points) Let $\Omega \subset \mathbb{R}^3$ be bounded and open and $0 < \tau_1 < \tau(x) < \tau_2$. Consider the PDE in the weak sense, $\nabla \cdot A(x)\nabla u(x) + \lambda\tau(x)u(x) = f(x)$ for $f \in L^2(\Omega)$ where $A(x) \in \mathbb{R}^{3\times 3}$ is self-adjoint with eigenvalues bounded above and below by positive real numbers. Define $H_0^1(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$\|\psi\|_1^2 := \int_\Omega \nabla\psi(x) \cdot A(x)\overline{\nabla\psi(x)}dx.$$

Show there exists a weak solution in $H_0^1(\Omega)$ for all λ except possibly a countable set.

10. (20 points)
Consider the operator $L = (-\triangle + 1)^{-1} : L^2([0,1]^3) \to L^2([0,1]^3)$. Prove L is a compact operator.