Introduction to Applied Mathematics
Qualifying Exam, August 2024

The exam is 3 hours. Each problem is worth 20 points, and you can do a maximum of five problems for a total potential score of 100 points. Pick at least one from each category.

Recall that $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space.

Category I: Continuum Mechanics

1. Let $\mathcal{V}=\mathbb{R}^3$. Consider the linear elastic equation $\rho_0 \frac{\partial^2}{\partial t^2} u = \nabla \cdot [A(\nabla u)]$ for $A \in \mathcal{V}^4$ ($\mathcal{V}^4$ is the four-fold tensor product of $\mathcal{V}$) and u a smooth vector field and $\rho_0 > 0$ a scalar constant. Let $\lambda, \mu > 0$ be fixed. Suppose that, for all $H \in \mathcal{V}^2$, we have $A(H) = \lambda(\mathrm{tr}\ H)I + \mu(H + H^T)$.

a. **(10 points)** Derive the Navier equation $\rho_0 \frac{\partial^2}{\partial t^2} u = \mu \triangle u + (\lambda + \mu)\nabla(\nabla \cdot u)$.

b. **(10 points)** Assume the PDE is defined over $\mathbb{R}^3$. Derive the solution for $u(x,t)$ using Fourier analysis **(no need to prove it is a classical solution, you're only asked to calculate)**. Assume $u(x,0) = f(x)$ and $\partial_t u(x,0) = g(x)$, where f and g are smooth and compactly supported vector fields.

2. **(20 points)** Consider a body in material coordinates described by an open set $B \subset \mathbb{R}^3$. Let $x = \varphi(X,t) = \varphi_t(X)$ be the change from material coordinates X to spatial coordinates x, and let $\rho(x,t)$ define the density field in spatial coordinates. Assume conservation of mass holds. Prove that if $\Phi(x,t)$ is smooth and scalar-valued and $\Omega_t = \varphi_t(\Omega)$ is the time evolution of some open $\Omega \subset B$ that

$$\frac{d}{dt}\int_{\Omega_t} \Phi(x,t)\rho(x,t)dV_x = \int_{\Omega_t} \frac{d}{dt}\Phi(x,t)\ \rho(x,t)dV_x.$$

Here we denote $\frac{d}{dt}\Phi(x,t) := \frac{\partial}{\partial t}\Phi(\varphi(X,t),t)\Big|_{X=\varphi_t^{-1}(x)}$.

3. **(20 points)** Denote $\mathcal{V} = \mathbb{R}^3$ and $\mathcal{V}^2 = \mathcal{V} \otimes \mathcal{V}$ (3×3 matrices). Define $\Sigma : \mathcal{V}^2 \to \mathcal{V}^2$ by $\Sigma(A) = \mathrm{tr}(A)A$. Denote by $D\Sigma(A) \in \mathcal{V}^4$ the derivative of Σ at $A \in \mathcal{V}^2$.
Prove that, for all $B \in \mathcal{V}^2$,

$$D\Sigma(A)B = \mathrm{tr}(B)A + \mathrm{tr}(A)B.$$

Category II: Fourier Analysis

4. Consider the inhomogeneous Helmholtz equation $(1-\triangle)u(x) = f(x)$ for $x \in \mathbb{R}^d$. Here, $\triangle = \sum_{0\leq j\leq d} \frac{\partial^2}{\partial x_j^2}$.

a. (10 points) Suppose $f \in \mathcal{S}(\mathbb{R}^d)$. Find $u(x)$ that solves the PDE such that $u(x) \to 0$ as $|x| \to \infty$.

b. (10 points) Suppose $f_n \in \mathcal{S}(\mathbb{R}^d)$ and $f_n \to f$ in the $L^2(\mathbb{R}^d)$ norm sense (that is, $\|f_n - f\|_2 \to 0$ as $n \to \infty$). Suppose u_n is a solution to $(1-\triangle)u_n(x) = f_n(x)$. Show $u_n \to u$ in the L^2 sense, and show u satisfies $\langle (1-\triangle)\psi, u\rangle = \langle \psi, f\rangle$ for all $\psi \in \mathcal{S}(\mathbb{R}^d)$.

5. (20 points) Consider the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ defined by $\mathcal{F}\psi(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}\psi(x)dx$. Prove if $\phi(x) = e^{-\pi x^2}$ that $\mathcal{F}\phi(\xi) = \phi(\xi)$.

6. (20 points) Consider the PDE $-\nabla\cdot A\nabla u(x) + u(x) = f(x)$ for $f \in C^\infty_{\text{per}}([0,1]^3)$ where $A = A^*$ is a constant 3×3 matrix. Suppose A has eigenpairs $(v_i, \lambda_i)_{i=1}^3$ with eigenvalues $\lambda_i > 0$ for all i. Find the solution $u \in C^\infty_{\text{per}}([0,1]^3)$. Prove the smoothness of u.

Category III: Weak-form PDEs & Distribution Theory

7. (20 points) Prove that the Dirac delta-function is not equal as a distribution to any continuous function.

8. Define the Fourier transform of $\psi \in \mathcal{S}(\mathbb{R}^d)$ by $\mathcal{F}\psi(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}\psi(x)dx$.

a. (10 points) Consider the delta-function as a tempered distribution $\delta \in \mathcal{S}'(\mathbb{R}^d)$. Find $\mathcal{F}\delta$.

b. (10 points) Find $\mathcal{F}(\mathcal{F}\delta)$.

9. (20 points) Define the space H_1 to be the closure of smooth functions u on $[0,1]$ such that $u(0) = u(1) = 0$ with respect to the norm $\|f\|_{H^1} = \|\frac{d}{dx}f\|_2$ where $\|\cdot\|_2$ denotes the standard $L^2([0,1])$ norm. Consider the inclusion operator $\iota : H_1 \to L^2([0,1])$ defined by $\iota f = f$. Show ι is a compact operator, i.e. there is a sequence of finite rank operators ι_n such that $\iota_n \to \iota$ in operator norm. (This is a 1D Rellich-Kondrachov Theorem).