Qualifier Exam **Core Analysis** **January 2025**

There are twelve problems below in three parts.
Complete FIVE problems, doing at least ONE problem from each part.
All problems have the same weight of twenty points. Please mark the problems you want to be graded, and make sure to have your name clearly written on the solution sheets. You can use "well known theorems" from lectures or any standard book on measure and integration, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied, otherwise you might not get full credit.

Part A:
(1) A bounded function F is said to be of bounded variation on $\mathbb{R}$ if F is of bounded variation on any finite subinterval $[a,b]$ and $\sup_{[a,b]} T_F(a,b) < \infty$. Prove that, for such an F, there is some constant A such that for all $h \in \mathbb{R}$,

$$\int_{\mathbb{R}} |F(x+h) - F(x)| dx \leq A|h|.$$

(2) From the polynomial $p(x) = x^3 - x^2$ we construct the function $f(x) = |p(x)|^{-2/3}$, which is measurable and defined almost everywhere on $\mathbb{R}$. Show that f is not integrable on $\mathbb{R}$. Show that f is integrable on $(-\infty, -1/n] \cup [1/n, \infty)$. Does there exist a largest subset $S \subsetneq \mathbb{R}$ such that f is integrable on S?

(3) Let $f \in L^1(\mathbb{R})$ with respect to Lebesgue measure, and suppose that

$$\int_{\mathbb{R}} |x||f(x)| dx < \infty.$$

Show that the function

$$g(y) := \int_{\mathbb{R}} \cos(xy) f(x) dx$$

is differentiable at every $y \in \mathbb{R}$.

(4) Define the function

$$f_n(x) = \mathbb{1}_{[-n,n]} \frac{\sin(n^2x^2)}{n^2|x|^{3/2}} = \begin{cases} \dfrac{\sin(n^2x^2)}{n^2|x|^{3/2}} & \text{if } 0 \leq |x| \leq n, \\ 0 & \text{if } |x| > n. \end{cases}$$

a) Prove that each f_n is integrable, despite the singularity at $x = 0$.
b) Compute (and cite the relevant convergence theorem) the limit:

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) dx$$

HINT: It is simultaneously true that $|\sin \theta| \leq |\theta|$ and that $|\sin \theta| \leq 1$.

Part B:

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function that is not almost everywhere infinite. Prove that there exists a subset $S \subset \mathbb{R}$ of positive measure such that f is bounded on S.

(2) Let $A \subset \mathbb{R}$ be a set of finite Lebesgue measure. Let $f_n \to f$ pointwise on A.

a) State the conclusion of Egorov's Theorem for the sequence $\{f_n\}_n$ and the set A.

b) Show by example that the conclusion of Egorov's Theorem might not hold if $m(A) = \infty$.

(3) Prove that, for every Lebesgue measurable function f on $\mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\sin(nx)dx = 0.$$

HINT: There are several nice families of functions that are dense in L^1.

(4) Let $E_1 \supset E_2 \supset ... \supset E_n \supset ...$ be a decreasing sequence of sets in the complete measure space $(X, \mathcal{M}, \mu)$. Assume that $\mu\left(\cap_{n=1}^{\infty} E_n\right) = 0$. If f is μ-integrable, prove that

$$\lim_{n\to\infty} \int_{E_n} f(x)d\mu(x) = 0.$$

HINT: Use Radon-Nikodym for the measure ν given by $\nu(E) = \int_E f d\mu$.

Part C:

(1) If f and g are two functions in $L^2(\mathbb{R}^d)$ and $|x_n| \to \infty$ as $n \to \infty$, show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)g(x + x_n)dx = 0.$$

(2) For a given $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, we denote by f^* its maximal function.

a) Prove that $\|f^*\|_{L^\infty} \leq \|f\|_{L^\infty}$.

b) For $1 < p < \infty$ and every x, prove that $(f^*(x))^p \leq (|f|^p)^*(x)$.

(3) Let $f \in L^p([0,1])$ for some $1 < p \leq \infty$; so in particular it is integrable on $[0,1]$. Define $F(x) := \int_0^x f(y)dy$, so that $F'(x) = f(x)$ (a.e.). We say that G is $C^\alpha([0,1])$ for $\alpha \in (0,1]$ if

$$\sup_{x,y\in[0,1]} \frac{|G(x) - G(y)|}{|x - y|^\alpha} < \infty.$$

Prove that F is $C^{1-1/p}([0,1])$.

(4) For f and g locally integrable, recall that $(f * g)(x) := \int f(x - y)g(y)dy$.

a) State Young's Inequality for Convolutions.

b) If $0 \leq f(x) \leq 1$ and $f \in L^q$ for some $q \geq 1$, how many times would you have to convolve f with itself to return an essentially bounded function?