
There are twelve problems below in three parts.
Complete FIVE problems, doing at least ONE problem from each part.
All problems have the same weight of twenty points. Please mark the problems you want to be graded, and make sure to have your name clearly written on the solution sheets. You can use "well known theorems" from lectures or any standard book on measure and integration, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied, otherwise you might not get full credit.

Part A:

(1) Suppose, for all $h > 0$, we have a measurable $E_h \subset B_h(0) \subset \mathbb{R}^d$ such that $m(E_h) \geq \frac{1}{2}m(B_h(0))$. If $f \in L^1_{\text{loc}}$ and 0 is a Lebesgue point for f, prove that

$$\lim_{h\to 0+} \frac{1}{m(E_h)} \int_{E_h} f(x)dx = f(0).$$

(2) Let $f \in L^1(0,\infty)$ and suppose that $\int_0^\infty x|f(x)|dx < \infty$. Prove that the function

$$g(y) := \int_0^\infty e^{-xy} f(x)dx$$

is differentiable at every $y \in (0,\infty)$. That is,

$$\lim_{h\to 0} \frac{g(y+h) - g(y)}{h} \quad \text{exists.}$$

(3) Let $\chi_{[-n,n]}(\cdot)$ denote the characteristic function of the interval $[-n,n]$ for $n \in \mathbb{N}$. Consider the sequence of functions $f_n(x) := \chi_{[-n,n]}(x)\sin(\pi x/n)$, for $x \in \mathbb{R}$.

a) Determine the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ and show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on any fixed compact subset of $\mathbb{R}$. Does the sequence converge uniformly on $\mathbb{R}$?

b) Show that

$$\int_{\mathbb{R}} f(x)dx = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx.$$

Are the assumptions of the Lebesgue dominated convergence theorem satisfied?

(4) From the polynomial $p(x) = x^3 - 4x^2 + 4x$, we construct the function $f(x) = 1/\sqrt{|p(x)|}$ which is measurable and defined almost everywhere on $\mathbb{R}$. Show that f is not integrable on $\mathbb{R}$. Show that f is integrable on $(-\infty, 2-1/n] \cup [2+1/n, \infty)$. Does there exist a largest subset $S \subsetneq \mathbb{R}$ such that f is integrable on S?

Part B:

(1) Let $A \subset (0,1)$ be a measurable set with $m(A) = 0$. Show that

$$m(\{x^2 : x \in A\}) = 0 \quad \text{and} \quad m(\{\sqrt{x} : x \in A\}) = 0.$$

(2) Define $\mathcal{A} = \{\emptyset, [0,1], [0,1]^C, \mathbb{R}\}$, and the premeasure μ^0 by

$$\mu^0(\emptyset) = 0, \quad \mu^0([0,1]) = 1, \quad \mu^0([0,1]^C) = \mu^0(\mathbb{R}) = \infty.$$

a) For an arbitrary $E \subset \mathbb{R}$, what is $\mu^*(E)$?
b) What are the μ^*-Caratheodory measurable subsets of $\mathbb{R}$?

(3) Suppose f is nonnegative and integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x : f(x) > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty m(E_\alpha)d\alpha.$$

(4) Let $E_1,\ E_2,\ \ldots,\ E_n$ be measurable subsets of $[a,b]$. Assume that each point $x \in [a,b]$ lies in at least m of these subsets. Prove that there exists $k \in \{1,\ldots,n\}$ such that $m(E_k) \geq \frac{m}{n}(b-a)$.
HINT: This can be solved with Fubini's theorem.

Part C:

(1) Let Ω be a measure space. Assume that $f \in L^1(\Omega)$ and also $f \in L^\infty(\Omega)$.
a) Prove that for every $p \in (1,\infty)$, $f \in L^p(\Omega)$.
b) Prove that $\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$.

(2) Let X be the normed linear space obtained by putting the norm $\|f\| = \int_0^1 |f(t)|dt$ on the set of real-valued continuous functions on $[0,1]$.
a) Show that X is not a Banach space.
b) Show that the linear functional $\Lambda f := f(1/2)$ is not bounded ($\Lambda : X \to \mathbb{R}$).

(3) Let $1 < p < \infty$ and $1 \leq q \leq \infty$ with $p \neq q$. Give an example of a measurable function f on $\mathbb{R}$ such that $f \in L^p(\mathbb{R})$ but $f \notin L^q(\mathbb{R})$.

(4) Let f and g be functions in $L^2(\mathbb{R}^d)$. Suppose that $|x_n| \to \infty$ as $n \to \infty$. Show that

$$\int_{\mathbb{R}^d} f(x)g(x+x_n)dx \to 0 \quad \text{as} \quad n \to \infty.$$