ALGEBRA QUALIFYING EXAM, JANUARY 2025

Do 5 of the following problems, including at least one from each of parts A, B, and C. The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have three hours. Good luck!

Part A

1. Let $(\mathbb{Z}/n\mathbb{Z})^*$ be the group of units in the ring $\mathbb{Z}/n\mathbb{Z}$. We write elements of $\mathbb{Z}/n\mathbb{Z}$ as cosets
$$[i]_n := i + n\mathbb{Z}.$$
   (a) Find the $[i]_{15}$ such that $[i]_{15}[7]_{15} = [1]_{15}$ in $(\mathbb{Z}/15\mathbb{Z})^*$.
   (b) How many elements does $G = (\mathbb{Z}/27\mathbb{Z})^*$ have?
   (c) Determine the structure of G using the classification of finitely generated abelian groups.
2. Let G be a group with 21 elements.
   (a) Show that G has a unique subgroup N of order 7, and that N is a normal subgroup.
   (b) Show that G has a subgroup K of order 3. Is K necessarily a normal subgroup? Explain.
   (c) From part (a), we have an exact sequence
$$1 \longrightarrow N \longrightarrow G \xrightarrow{p} G/N \longrightarrow 1.$$
   Is it always true that we have an isomorphism $G \cong N \times G/N$? Explain.
3. (a) Define what it means for a group G to act on a set X.
   (b) If the group S_5 acts on itself by conjugation ($X = S_5$), describe the orbits. How many are there?
   (c) If S_5 acts on itself by conjugation, then it acts on the set Y of subgroups of order 8 in S_5. How many orbits does S_5 have on Y?. Note $5! = 8 \cdot 15$. Describe a group in each orbit.

Part B

1. Let $I, J \subset \mathbb{Z}[X]$ be the ideals $I = \langle 7, X^2 + 1\rangle$, $J = \langle 7, X^2 + 3\rangle$.
   (a) Show that there is a ring isomorphism
$$\mathbb{Z}[X]/J \cong \mathbb{F}_7[X]/\langle X^2 + 3\rangle \cong \mathbb{F}_7 \times \mathbb{F}_7.$$
   Is J a prime ideal? Explain. Hint: Factor $X^2 + 3$ mod 7.
   (b) Show that $X^2 + 1$ is irreducible mod 7. Explain why this shows that $\langle X^2 + 1\rangle$ is a maximal ideal in $\mathbb{F}_7[X]$. Then explain why $\mathbb{Z}[X]/I$ is a field. How many elements does it have?
2. (a) Let $\varphi : B \to A$ be a ring homomorphism (ring = ring with $1 \neq 0$; homomorphisms take 1 to 1). Show that: if $P \subset A$ is a prime ideal, then $\varphi^{-1}(P) \subset B$ is a prime ideal.
   (b) Give an example to show that if $M \subset A$ is a maximal ideal, then $\varphi^{-1}(M) \subset B$ need not be a maximal ideal.
   (c) Describe $\mathrm{Spec}(\mathbb{C}[X])$.
3. Let $f(X, Y) = Y^2 - X^5 \in \mathbb{Q}[X, Y]$.
   (a) Show that $f(X, Y)$ is irreducible in $\mathbb{Q}[X, Y]$.

(b) Explain why part (a) shows that the principal ideal $\langle Y^2 - X^5 \rangle$ is a prime ideal in $\mathbb{Q}[X,Y]$. Hint: $\mathbb{Q}[X,Y]$ is a UFD.
(c) Let $\varphi : \mathbb{Q}[X,Y] \to \mathbb{Q}[T]$ be the $\mathbb{Q}$-algebra homomorphism such that $\varphi(X) = T^2$, $\varphi(Y) = T^5$. Show that $\ker(\varphi) = \langle Y^2 - X^5 \rangle$.

Part C

1. Let $A, B \in M_n(F)$ be square matrices of size n with entries in a field F.
   (a) Describe an algorithm to decide whether A and B are similar. That is, whether there is an invertible matrix $P \in GL_n(F)$ such that $B = PAP^{-1}$.
   (b) Let
   $$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 4 & 3 & -2 & 0 \\ -2 & 1 & 5 & 0 \\ 2 & 0 & -1 & 3 \end{pmatrix}.$$
   The Smith form of $xI_4 - A$ is $\mathrm{diag}(1, 1, x-3, (x-3)^3)$. Write down the rational canonical form C of A, and the Jordan canonical form J of A. Are there matrices $P, Q \in GL_4(\mathbb{Q})$ such that $C = PAP^{-1}$ and $J = QAQ^{-1}$? Explain.
2. Let R be a ring (with $1 \neq 0$). Recall that the set
   $$\mathrm{Hom}_R(M,N) = \{f : M \to N \mid f \text{ is an } R-\text{module homomorphism}\}$$
   has the natural structure of an abelian group (here: module = left module).
   (a) If $u : M_1 \to M_2$ and $v : N_1 \to N_2$ are R-module homomorphisms, show that we have abelian-group homomorphisms
   $$\mathrm{Hom}_R(u,N) : \mathrm{Hom}_R(M_2,N) \to \mathrm{Hom}_R(M_1,N)$$
   $$\mathrm{Hom}_R(M,v) : \mathrm{Hom}_R(M,N_1) \to \mathrm{Hom}_R(M,N_2).$$
   (b) If we fix M (resp. N) the assignment
   $$N \mapsto \mathrm{Hom}_R(M,N), \quad (\text{resp. } M \mapsto \mathrm{Hom}_R(M,N))$$
   is a mapping from the class of R-modules to the class of Abelian Groups. Explain why this is a covariant (resp. contravariant) functor.
3. Let R be a commutative ring.
   (a) Define what it means for an R-module M to be Noetherian.
   (b) State Hilbert's Basis Theorem.
   (c) If
   $$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
   is an exact sequence of R-modules, show that M is Noetherian if and only if both M' and M'' are Noetherian. In other words, if $M' \subset M$ is a submodule and $M'' = M/M'$ is the quotient, then M is Noetherian if and only if M' and M'' are.