ALGEBRA QUALIFYING EXAM

August 2024

Answer five of the following ten questions, including at least one from each of parts I, II, and III.

Part I

(1) Let G be a finite group, and let Z denote the center of G.
   (a) Show that Z is a normal subgroup of G.
   (b) Show that if G/Z is cyclic, then G must be abelian.
   (c) Find the center of the symmetric group S_3.

(2) Let $\varphi : G \to H$ be a surjective group homomorphism, and let N be a normal subgroup of G. Show that $\varphi(N)$ is a normal subgroup of H. What happens if φ is not surjective?

(3) Let R be an integral domain, and let G be a finite subgroup of $R^\times$, the group of units of R. Prove that G is cyclic.

(4) Suppose G is a simple group of order 168. How many elements of order 7 does it contain?

Part II

(5) Let R be a PID, and let I and J be nonzero ideals in R. Show that $IJ = I \cap J$ if and only if $I + J = R$.

(6) In the ring $\mathbb{Z}[x]$, consider the ideals $I = \langle x^2 + 2, 5 \rangle$ and $J = \langle x^2 + 2, 3 \rangle$. Show that I is a maximal ideal, and that J is *not* a maximal ideal.

(7) Let R be a finite commutative ring with 1. Show that any prime ideal in R is a maximal ideal.

Part III

(8) Let R be a ring, and let $f : M \to N$ be a surjective homomorphism of R-modules. Assume that N is a free R-module. Show that there exists an R-module homomorphism $g : N \to M$ such that $f \circ g$ is the identity map $N \to N$. Show also that $M = \ker(f) \oplus \operatorname{im}(g)$.

(9) Let V be an n-dimensional vector space over a field, and let $N : V \to V$ be a linear transformation. Suppose that $N^n = 0$, but $N^{n-1} \neq 0$. Prove that there is no linear transformation $T : V \to V$ such that $T^2 = N$.

(10) Prove that any $n \times n$ complex matrix A is similar to transpose A^{t}.