Print Your Name Here: ____________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 200.

Part I: Short Questions. Answer **12** of the 18 short questions: 8 points each. **Circle** the **numbers** of the 12 questions that you want counted—*no more than 12*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 96 points.

1. If $f \in \mathcal{R}[0,1]$, use the Cauchy-Schwarz inequality to find a constant K such that
$\int_0^1 xf(x)\,dx \le K \left[\int_0^1 [f(x)]^2\,dx\right]^{\frac{1}{2}}$.

2. True or Give a Counterexample: If f is *differentiable* and *uniformly continuous* on $[a,b]$ then the derivative, $f'(x)$, must be *bounded* on $[a,b]$.

3. True or Give a Counterexample: If f is monotone increasing on $[0,1]$ and monotone decreasing on $[1,2]$, then $f \in \mathcal{R}[0,2]$.

4. Let $f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x \le \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < x \le 1 \end{cases}$ for each $n > 1$. Find

a. $\lim\limits_{n\to\infty} f_n(x)$ for each $x \in [0,1]$.

b. $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx$.

5. Let $T : \mathcal{C}[0,1] \to \mathbb{R}$ be defined by $T(f) = \displaystyle\int_0^1 f(x)\left(1+x^2\right)\,dx$. Find a constant K for which $|T(f)| \le K\|f\|_{\sup}$ for all $f \in \mathcal{C}[0,1]$.

6. Use the *triangle inequality* for $\|\cdot\|_2$ to find the value of K for which
$$\left[\int_0^{\frac{\pi}{2}} (\sqrt{\cos x} + x)^2\,dx\right]^{\frac{1}{2}} \le 1 + K$$

7. True or Give a Counterexample: If a sequence of differentiable functions f_n converges uniformly on $\mathbb{R}$ to a differentiable function g, then $f_n'(x)$ converges to $g'(x)$.

8. Let $V = \left\{ x \,\middle|\, \displaystyle\sum_{k=1}^{\infty} x_k \text{ converges} \right\}$ be the vector space of summable sequences. True or Give a Counterexample: $T : V \to \mathbb{R}$ by $T(x) = \displaystyle\sum_{k=1}^{\infty} x_k$ is *linear*.

9. Let $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$. Find the set $\{x \mid f'(x) \text{ exists}\}$.

10. Give an example of a sequence f_n for which $f_n' \to 0$ uniformly on $\mathbb{R}$, yet $f_n(x)$ *diverges* for all $x \in \mathbb{R}$.

11. Let p be a polynomial of degree n and let $E = \{x \mid e^x = p(x)\}$. What is the largest number of elements that there can be in the set E?

12. Find $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$. (Hint: Rewrite the kth term to see that this is an example of a *telescoping series.*)

13. For all $n \in \mathbb{N}$ define a sequence $x^{(n)} \in l_1$ by letting $x_k^{(n)} = \dfrac{n+1}{n2^k}$, for all $k \in \mathbb{N}$. Find $\left\|x^{(n)}\right\|_1$.

14. Does $\sum_{k=1}^{\infty} \sin^k x$ converge *uniformly* on $\left[0, \frac{\pi}{2}\right)$?

15. True or Give a Counterexample: If x_j is a *conditionally* summable sequence and if y_k is *not* identically 0, then the countable family of terms $x_j y_k$ is *divergent* in some order.

16. If $\sum_{k=1}^{\infty} f_k$ converges uniformly on D, find $\lim\limits_{n\to\infty} \|f_n\|_{\text{sup}}$.

17. Let $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ on $\mathbb{R}$. True or Give a Counterexample: $f'(x)$ exists and equals $\displaystyle\sum_{k=1}^{\infty} \frac{\cos kx}{k}$ for all $x \in \mathbb{R}$.

18. True or False: $\displaystyle\sum_{k=1}^{\infty} \frac{\sin kx}{k^2} \in \mathcal{C}(\mathbb{R})$.

Part II: Proofs. *Prove carefully* **4** of the following 6 theorems for 26 points each. **Circle** the *letters* of the 4 proofs to be counted in the list below—*no more than 4!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.

A. If f and g are in $\mathcal{R}[a,b]$, we say f is *orthogonal* to g, denoted by $f \perp g$, if and only if $\langle f, g\rangle = 0$. Prove that $f \perp g \Leftrightarrow \|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$. (Here $\langle f, g\rangle = \displaystyle\int_a^b f(x)g(x)\,dx$. This is a modern analogue of the *Pythagorean Theorem.*)

B. Let $g \in \mathcal{R}[a,b]$, and define $T_g : \mathcal{C}[a,b] \to \mathbb{R}$ by $T_g(f) = \displaystyle\int_a^b f(x)g(x)\,dx$ for all $f \in \mathcal{C}[a,b]$. Prove: T_g is a bounded linear functional on the Banach space $\mathcal{C}[a,b]$, equipped with the sup-norm. Remember to find a suitable *bound* for T_g, and be explicit about theorems that you use.

C. Prove that e is irrational. (Hint: Suppose false, so that $e = \frac{p}{q}$, where $p, q \in \mathbb{N}$. Write $e = e^1 = P_n(1) + R_n(1)$, multiply both sides by $n!$, and deduce a contradiction when $n \in \mathbb{N}$ is sufficiently large. Here P_n and R_n are the nth Taylor Polynomial and Remainder, respectively.)

D. Let $f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^3}$ for all $x \in \mathbb{R}$. Prove that $f \in \mathcal{C}^1(\mathbb{R})$ and find an expression for $f'(x)$ in terms of an infinite series. Be sure to show clearly what theorem(s) you are using and that you have checked all the necessary hypotheses.

E. We have shown in class and in our text that if we let $g(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$ then

$\tan^{-1}(x) = g(x)$ for all $x \in (-1, 1)$.

- **(i)** (10) Prove that the series for $g(x)$ converges *uniformly* on the *closed* interval $[-1, 1]$ and that $g \in \mathcal{C}[-1, 1]$. (Hint: Use the alternating series test.)
- **(ii)** (10) Use the preceding result to show that $\tan^{-1}(x) = g(x)$ remains valid for all $x \in [-1, 1]$. (Hint: Use continuity.)
- **(iii)** (6) Use the preceding result to find an infinite series the sum of which is π. (Hint: What is $\tan^{-1}(1)$?)

F. Let $F(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } 0 < |x| \leq 1, \\ 0 & \text{if } x = 0 \end{cases}$ and let $f(x) = F'(x)$. (See Fig. 1.)

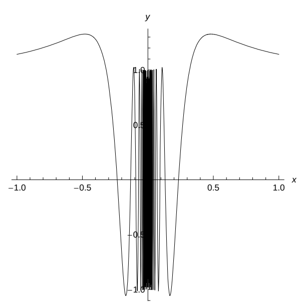

Figure 1: $y = f(x)$.

- **(i)** (10) Prove that $F'(x)$ exists for all $x \in [-1, 1]$. (Be sure to include $x = 0$.)
- **(ii)** (10) Find $f(x)$ for all $x \in [-1, 1]$, and prove $f \in \mathcal{R}[-1, 1] \setminus \mathcal{C}[-1, 1]$. (Hint: You may quote a previous exercise or theorem from our course.)
- **(iii)** (6) Find $\int_{-1}^{1} f(x)\, dx$.

Solutions and Class Statistics

1. $K = \dfrac{1}{\sqrt{3}}$.

2. Counterexample: Let $F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0 \end{cases}$ on $[0,1]$. F is continuous, hence uniformly continuous, on $[0,1]$, but $F'(x)$ is unbounded as $x \to 0+$, although $F'(0)$ exists and is zero.

3. True

4.

a. $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0,1]$.

b. $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \dfrac{1}{2}$.

5. $K = \dfrac{4}{3}$ works.

6. $K = \sqrt{\dfrac{\pi^3}{24}}$

7. Counterexample: Let $f_n(x) = \dfrac{1}{n} \sin\left(n^2 x\right)$. Then the sequence $f_n'(0)$ diverges. See problem 4.35.

8. True

9. $\{0\}$ since $f'(0)$ exists and is zero. But $f'(x)$ does not exist, indeed f is not even continuous at x, if $x \ne 0$.

10. For example, let $f_n(x) = n$ for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$.

11. $n + 1$

12. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$

13. $\left\|x^{(n)}\right\|_1 = \dfrac{n+1}{n}$. See exercise 5.36.

14. No. If this series did converge uniformly, its partial sums would be bounded in the sup-norm, but they are not.

15. True, since the countable family fails to be absolutely convergent in any order.

16. 0

17. Counterexample: The differentiated series diverges for $x = 0$.

18. True, since the series converges uniformly by the Weierstrass M-test.

Remarks about the proofs

Proofs are graded for logical coherence. Be sure to state what is your hypothesis (the assumption) and what conclusion you are seeking to prove. Then include justifications for each step. Your job is to show me through your writing that you understand the reasoning. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: The main difficulty encountered by some students was the distinction between a proposition and its converse. The proof needs cover both directions of implication.

B: The product $fg \in R[a, b]$, being a product of two Riemann integrable functions. This refers to a theorem proven in class and in the text just before the Cauchy-Schwarz inequality. It is easy to check that T_g is linear. Since $T_g(f) \in \mathbb{R}$, we need $K \in [0, \infty)$ such that $|T_g(f)| \leq K\|f\|_{\text{sup}}$. Using the triangle inequality for integrals, we find that $K = \int_a^b |g(x)|\,dx$.

C: Be sure to follow the hint and show how you make use of the denominator q and its relation to the choice of n when you multiply by $n!$. Some students made this way too complicated by ignoring the hint and using an infinite series for e, and making an unjustified claim about the sum of an infinite series being irrational.

D: Be sure to say what theorem you are invoking: what are all its hypotheses and conclusions, and how you check these in order to claim that $f'(x)$ exists and equals the sum of the term-by-term derivatives.

E: Be sure to show how you use the error estimate from the Alternating Series Test to establish uniform convergence on $[-1, 1]$. Remember to check both the alternating property and monotonicity of convergence to zero for the Alternating Series Test. Use continuity of both g amd $\tan^{-1}$ on $[-1, 1]$.

F: Find $f'(x)$ for all $x \in [-1, 1]$. You can prove $f \in R[-1, 1]$ using exercise 3.26 from your homework. Be sure to prove discontinuity of f at 0. No hand-waving. Then use the Fundamental Theorem of Calculus–the version not requiring f to be continuous.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	5	2	5	5	6
80-89 (B)	3	3	2	1	2
70-79 (C)	1	1	2	2	1
60-69 (D)	0	3	0	1	0
0-59 (F)	0	0	0	0	0
Test Avg	89.6%	79.4%	88.3%	85.45%	86.1%
HW Avg	86.8%	87.6%	82.1%	81.5%	81.5%
HW/Test Correl	0.45	0.63	0.68	0.68	0.68

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^9$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework. (In my experience, this statistic is somewhat unstable in classes with low enrollment.)