Print Your Name Here: ____________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart phones, cell phones, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 200.

Part I: Short Questions. Answer **12** of the 18 short questions: 8 points each. **Circle** the **numbers** of the 12 questions that you want counted—*no more than 12*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 96 points.

1. Suppose for all $n \in \mathbb{N}$ we have $y_n \neq 0$. True or else Give a Counterexample: If both $x_n y_n$ and $\frac{x_n}{y_n}$ converge, then x_n converges and y_n converges.

2. Let $E \subseteq \mathbb{R}$ be any *unbounded* set. Find an open cover $\mathcal{O} = \{O_n \mid O_n \text{ is open } \forall n \in \mathbb{N}\}$ of E that has no finite subcover.

3. True or False: The sequence x_n described as: $\mathbf{0}, \mathbf{1}, \frac{1}{2}, \mathbf{0}, \frac{1}{3}, \frac{2}{3}, \mathbf{1}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \mathbf{0}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \mathbf{1}, \ldots$ is a Cauchy Sequence.

4. Suppose $s_n \leq t_n \leq u_n$ for all n, $s_n \to a < b$, and $u_n \to b$. True or Give a Counterexample: $\lim_{n\to\infty} t_n \in [a, b]$.

5. Let $x_n = (-1)^n + \frac{1}{n}$. Find $\limsup x_n$ and $\liminf x_n$. Does $\lim_{n\to\infty} x_n$ exist?

6. True or Give a Counterexample:

a. A bounded sequence times a convergent sequence must be convergent.

b. A null sequence times a bounded sequence must be a null sequence.

7. Let $f \in \mathcal{C}(\mathbb{R})$ such that $f(x+y) \equiv f(x) + f(y)$. If $f(2) = 5$, find $f(7)$.

8. Let $f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$

a. True or False: $f \notin \mathcal{C}[0,1]$ but does have the *intermediate value property* on $[0,1]$.

b. True or False: $f \in \mathcal{C}(0,1]$ but f is *not uniformly* continuous on $(0,1]$.

9. Give an example of a decreasing nest of *infinite* interval*s* with empty intersection.

10. Find an open cover of the interval $(-1,1)$ that has no finite subcover.

11. Give an example of a function ϕ that maps the set of all natural numbers one-to-one and onto the set of all even natural numbers.

12. True or False: An open dense subset of $\mathbb{R}$ must be all of $\mathbb{R}$.

13. Find the set of all cluster points of the set $\mathbb{Q} \cap (0,1)$.

14. Let $f(x) = \begin{cases} 1-x & \text{if } x \in \mathbb{Q}, \\ 1-x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find the set of all points at which f is continuous.

15. Suppose $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{(-1)^n}{n} & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right],\ n \in \mathbb{N} \end{cases}$. True or False: $f \in \mathcal{R}[0,1]$

16. Let $f(x) = \begin{cases} \sin\frac{\pi}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$ True or False: $f \in \mathcal{R}[0,1]$.

17. Let $f_n(x) = \dfrac{x^2}{x^2+n}$ for each $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$.

a. Find the $f(x) = \lim\limits_{n\to\infty} f_n(x)$, the pointwise limit.

b. Find $\|f_n - f\|_{\sup}$. Is the convergence uniform?

18. Express $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(k\frac{1}{n}\right)^2$ as a definite integral.

Part II: Proofs. *Prove carefully* **4** of the following 6 theorems for 26 points each. **Circle** the *letters* of the 4 proofs to be counted in the list below—*no more than 4!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.

A. Suppose $a \leq x_n \leq b$ for all $n \in \mathbb{N}$ and suppose further that $x_n \to L$. Prove: $L \in [a, b]$. (Hint: Prove there is a contradiction if $L < a$, and also if $L > b$.)

B. A function f is called monotone increasing provided whenever $x_1 < x_2$ in D_f we have $f(x_1) \leq f(x_2)$. Prove: If f is monotone increasing on $\mathbb{R}$, then for all $a \in \mathbb{R}$, $\lim_{x \to a+} f(x)$ exists and is a real number. (Hint: Let $S = \{f(x) \mid x > a\}$ and show that $\inf(S)$ is a real number L. Then show that for every sequence $x_n \to a+$ we must have $f(x_n) \to L$.)

C. Prove the following *fixed point theorem*: Suppose $f \in \mathcal{C}[0,1]$ and suppose $0 \leq f(x) \leq 1$ for all $x \in [0,1]$. Then there exists $c \in [0,1]$ such that $f(c) = c$. (Hint: Consider $g(x) = f(x) - x$.)

D. Let $f_n(x) = 1 - x^n$ for each $n \in \mathbb{N}$. Decide whether or not f_n *converges uniformly* on each interval below, and *prove* your conclusion.

- **(i)** (6) $[0, 1]$
- **(ii)** (10) $[0, b]$, where $0 \leq b < 1$
- **(iii)** (10) $[0, 1)$

E. Let $f \in \mathcal{C}[a, b]$. Prove: There exists $\bar{x} \in [a, b]$ such that $\int_a^b f(x)\, dx = f(\bar{x})(b - a)$ Be sure to explain which theorem(s) you are using and why their hypotheses are satisfied.

F. Suppose $f \in \mathcal{C}(a, b)$ and also that f is bounded on $[a, b]$. Prove: $f \in \mathcal{R}[a, b]$. (Hint: Use the *Variant form* of the Darboux Integrability Criterion.)

Solutions and Class Statistics

1. Counterexample: Let $x_n = (-1)^n = y_n$.

2. For example, let $O_n = (-n, n)$ for all $n \in \mathbb{N}$. Do not confuse an open cover with the union of its member sets: the union is just one set.

3. False: It has infinitely many 0's and 1's, and it does not converge.

4. Counterexample: Let $s_n = -1,\ t_n = (-1)^n,\ u_n = 1$, for all $n \in \mathbb{N}$.

5. $\limsup x_n = 1$ and $\liminf x_n = -1$. The limit does not exist.

6.

a. Counterexample: Let $x_n = 1$ and $y_n = (-1)^n$ for all $n \in \mathbb{N}$.

b. True

7. $f(1) = \dfrac{5}{2}$, so $f(7) = \frac{35}{2}$. See Exercise 2.27.

8. True for both parts. See Exercise 2.47.

9. For example, let the nth interval be (n, ∞).

10. For example, let the open cover be the set of all the intervals $O_n = \left(\frac{1}{n} - 1, 1\right)$ for each $n \in \mathbb{N}$

11. For example, let $\phi(n) = 2n$.

12. False: for example, $(-\infty, 0) \cup (0, \infty)$ is dense and open but not equal to $\mathbb{R}$.

13. $[0, 1]$.

14. $\{0, 1\}$.

15. True

16. True.

17.

a. $f(x) = \lim\limits_{n\to\infty} f_n(x) \equiv 0$

b. $\|f_n - f\|_{\sup} = 1$ for all $n \in \mathbb{N}$. The convergence is not uniform.

18. $\displaystyle \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(k\frac{1}{n}\right)^2 = \int_0^1 x^2\,dx \left(= \frac{1}{3}\right).$

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: Most solutions to this problem were excellent. The most frequent error was failure to understand the definition of convergence of a sequence $x_n \in \mathbb{R}$ to a limit L. There was also some carelessness with inequalities.

B: There were many errors among the solutions offered. One was failure to distinguish between a lower bound of S and $\inf(S)$. It was important to understand that if $\epsilon > 0$ then $\inf(S) + \epsilon$ cannot be a lower bound for S. The rest of the proof unfolds from this fact, making proper use of monotonicity.

C: Most solutions were fine. Be careful with the two endpoint cases and with verifying the hypotheses of the Intermediate Value Theorem. It is important that all inequalities be correct.

D: Most solutions were very good. Most of the errors amounted to carelessness, making false claims or just leaving out the necessary reasons for the claims made.

E: It is important to note that f achieves its maximum and minimum values at points in the interval, so that the Intermediate Value Theorem can be applied.

F: The key is the judicious choice of the second point x_1 and the next-to-last point x_{n-1} in the partition of $[a, b]$, and the use of continuity on the closed finite interval $[x_1, x_{n-1}]$.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7	12	14	12	16
80-89 (B)	7	6	7	8	7
70-79 (C)	8	3	2	4	1
60-69 (D)	2	3	3	1	2
0-59 (F)	6	5	2	2	2
Test Avg	75.6%	79.8%	86.3%	86.15%	89%
HW Avg	7.2	7.62	7.4	7.2	7.2
HW/Test Correl	0.79	0.66	0.57	0.52	0.52

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{28}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a fairly strong positive correlation with performance on the homework.