Print Your Name Here: ____________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart phones, cell phones, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 200.

Part I: Short Questions. Answer **12** of the 18 short questions: 8 points each. **Circle** the **numbers** of the 12 questions that you want counted—*no more than 12*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 96 points.

1. True or Give a Counterexample: If $s_n \to L$ and $u_n \to M$, and if $s_n \le t_n \le u_n$ for all $n \in \mathbb{N}$, then $L \le \lim_{n\to\infty} t_n \le M$.

2. Give an example of a sequence x_n that has no convergent subsequence, yet it is false that x_n diverges either to ∞ or to $-\infty$.

3. True or False: The sequence x_n described as follows is Cauchy: $\mathbf{0}, \mathbf{1}, \frac{1}{2}, \mathbf{0}, \frac{1}{3}, \frac{2}{3}, \mathbf{1}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \mathbf{0}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \mathbf{1}, \ldots$.

4. Let $x_n = (-1)^n + \frac{1}{n}$. Find $\limsup x_n$ and $\liminf x_n$.

5. Find two sequences, x_n and y_n, for which $\limsup(x_n + y_n) < \limsup x_n + \limsup y_n$.

6. Find an open cover $\mathcal{O} = \{(a_n, b_n) \mid n \in \mathbb{N}\}$ of the interval $(-1, 1)$ that has *no finite subcover*.

7. Let f be a *monotone increasing* function on $\mathbb{R}$. True or False: $\lim_{x \to 0+} f(x) = f(0)$.

8. Give an example of a function f that is continuous on $(0, 1)$ without being *uniformly* continuous on $(0, 1)$.

9. True or False: an *open* set E that is also *dense* in $\mathbb{R}$ must be all of $\mathbb{R}$.

10. Find the *set* of all the *cluster points* of the set $\mathbb{Q}$ of all rational numbers.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + y) \equiv f(x) + f(y)$. If $f(1) = \sqrt{2}$, find $f(\sqrt{2})$.

12. Suppose f is *uniformly continuous* on a domain D and suppose $E \subset D$. True or False: f is uniformly continuous on E as well.

13. True or False: If a sequence of functions f_n *converges uniformly* on an interval I, then it must converge uniformly on every subinterval of I.

14. Let $\tilde{P}$ be the set of all functions f on $[0,1]$ such that there exists at least one point $x \in [0,1]$, perhaps depending on f, for which $f(x) > 0$. True or False: $\tilde{P}$ is a vector space.

15. True or False: There exists a function $f \in C[0,1]$ for which $\{f(x) \mid x \in [0,1]\} = (0,1)$.

16. Give an example of a *monotone increasing* function f on $[0,2]$ for which there is *no point* $\bar{x}$ such that $\int_0^2 f(x)\,dx = f(\bar{x})(2-0)$.

17. True or Give a Counterexample: The function $|f| \in \mathcal{R}[0,1] \Leftrightarrow f \in \mathcal{R}[0,1]$.

18. Give an example of a sequence $f_n \in R[0,1]$ which *converges pointwise* to 0, yet $\int_0^1 f_n(x)\,dx$ does *not* converge to 0.

Part II: Proofs. *Prove carefully* **4** of the following 6 theorems for 26 points each. **Circle** the *letters* of the 4 proofs to be counted in the list below—*no more than 4!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.

A. Suppose $a \leq x_n \leq b$ for all n and suppose further that $x_n \to L$. Prove: $L \in [a, b]$. (Hint: If $L < a$ or if $L > b$, obtain a contradiction.)

B. Suppose that the sequence x_n has *no* convergent subsequence. Let $M > 0$. Prove that there exist at most finitely many values of n such that $x_n \in [-M, M]$. Explain why this implies $|x_n| \to \infty$ as $n \to \infty$.

C. Let f be *monotone increasing* on $\mathbb{R}$: i.e., whenever $x_1 < x_2$ we have $f(x_1) \leq f(x_2)$. Prove: for all $a \in \mathbb{R}$, $\lim_{x \to a+} f(x)$ exists and is a real number. (Hint: Let $S = \{f(x) \mid x > a\}$ and show that $\inf(S)$ is a real number L. Then show that for every sequence $x_n \to a+$ we must have $f(x_n) \to L$.)

D. Let $f(x) = \dfrac{1}{x}$, for all $x \in (0, \infty)$. Prove, being *careful* to use the *definition* of *uniform continuity*:

(**i**) f is *uniformly* continuous on $(1, \infty)$.

(**ii**) f is *not* uniformly continuous on $(0, \infty)$?

E. Suppose $f_n(x) = 1 - x^n$ for all $x \in [0, 1)$. Being careful to use the *definition* of *uniform convergence*, prove:

(**i**) The sequence f_n *converges uniformly* on $[0, b]$ for all $b \in (0, 1)$.

(**ii**) The sequence f_n does *not* converge uniformly on $[0, 1)$.

F. If $f \in \mathcal{R}[a, b]$, prove the *triangle inequality for integrals*: $\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$. (Hint: Write the left side by expressing $f = f^+ - f^-$ and use the fact that f^+ and f^- are both nonnegative functions.)

Solutions and Class Statistics

1. Counterexample: Let $s_n = -1$, $t_n = (-1)^n$, and $u_n = 1$, for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty} t_n$ does not exist. It is necessary to define what is s_n and what is u_n for all $n \in \mathbb{N}$.

2. For example, let $x_n = (-1)^n n$ for all $n \in \mathbb{N}$.

3. False

4. $\limsup x_n = 1$ and $\liminf x_n = -1$.

5. For example, let $x_n = (-1)^n$ and $y_n = -x_n$.

6. For example, let $\mathcal{O} = \left\{ \left(-1, 1 - \frac{1}{n} \right) \mid n \in \mathbb{N} \right\}$.

7. False. For example, let $f(x) = 1_{(0,\infty]}(x)$ for all $x \in \mathbb{R}$.

8. For example, let $f(x) = \frac{1}{x}$.

9. False. For example, consider $(-\infty, 0) \cup (0, \infty)$.

10. $\mathbb{R}$.

11. $f(\sqrt{2}) = 2$, since $f(x) = cx$ and $f(1) = c$.

12. True

13. True

14. False: there is no additive identity, which would need to be the identically zero function.

15. False: f needs to achieve a maximum and a minimum value.

16. For example, let $f = 1_{[0,1]}$.

17. Counterexample: Let $f = 1_{\mathbb{Q}\cap[0,1]} - 1_{[0,1]\setminus\mathbb{Q}}$.

18. For example, let $f_n(x) = n 1_{\left(0,\frac{1}{n}\right]}(x)$.

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: The key to this proof is to understand clearly how the limit of a sequence is defined in terms of inequalities. If $L < a$ let $\epsilon = a - L$ which is strictly positive and show from the definition of convergence that for sufficiently large n, $x_n < a$. Do similar work if $L > b$, letting $\epsilon = L - b$ which is strictly positive. Most students did this well.

B: The key is to show that if there exists a subsequence $x_{n_k} \in [-M, M]$ for all $k \in \mathbb{N}$, then the Bolzano-Weierstrass theorem implies the existence of a sub-subsequence $x_{n_{k_j}}$ that converges as $j \to \infty$. This would contradict the hypothesis. Vague use of everyday language is not sufficient. Apply the Bolano-Weierstrass theorem clearly and correctly.

C: This was one of the harder collected homework problems, and only one student got it correctly on this test. The key steps are to explain that $f(a)$ is a lower bound for S, so that there is a greatest lower bound $L \geq f(a)$. (You should be able to give examples to show that $f(a)$ can be less than L.) If $\epsilon > 0$, $L + \epsilon$ is not a lower bound of S. You need to *use this fact* to show for *every* sequence $x_n \to a+$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|f(x_n) - L| < \epsilon$.

D: On $(1, \infty)$ show with simple inequalities that if $|x_1 - x_2| < \epsilon$ then $|f(x_1) - f(x_2)| < \epsilon$. Thus any $0 < \delta \leq \epsilon$ suffices for uniform continuity. However, on $(0, \infty)$, show that, for example, letting $\epsilon = 1$, for every $\delta > 0$ there exist $x_1, x_2 \in (0, \delta)$ such that $|f(x_1) - f(x_2)| > 1$. Thus f is not uniformly continuous on $(0, \infty)$.

E: Identify the pointwise limit $f(x)$ on $[0, b]$ and show that on $[0, b]$, $\|f_n - f\|_{\sup} = b^n \to 0$ as $n \to \infty$. But on $[0, 1)$ show that $\|f_n - f\|_{\sup} = 1$ which does not converge to 0.

F: Be careful with inequalities. Do not assume the conclusion as part of your proof!

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	2	3	4	4	6
80-89 (B)	6	4	2	3	4
70-79 (C)	2	2	4	2	1
60-69 (D)	1	2	2	2	1
0-59 (F)	1	1	0	1	0
Test Avg	81.3%	80.2%	79.4%	78.9%	80.7%
HW Avg	7.8	7.0	7.2	7.2	7.2
HW/Tst Correl	0.71	0.70	0.74	0.70	0.70

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{12}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.