Print Your Name Here: ______________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Give an example of three vectors, $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, for which $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$, showing your calculations of the products.

2. True or give a Counterexample: $x_n y_n$ converges if and only if x_n converges and y_n converges.

3. Give an example of a nonempty bounded set $S \subset \mathbb{R}$ for which $\sup(S) \notin S$.

4. True or False: Every bounded sequence of real numbers has a Cauchy subsequence.

5. True or False: A sequence $x_n \in \mathbb{R}$ is a Cauchy sequence if and only if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_{n+1} - x_n| < \epsilon$.

6. True or give a Counterexample: If $\dfrac{x_n}{y_n}$ converges and if $x_n y_n$ converges, then x_n converges and y_n converges.

7. Let $x_n = (-1)^n + \dfrac{1}{n}$. Find $\limsup x_n$ and $\liminf x_n$.

8. Give an example of two *bounded* sequences, x_n and y_n, for which $\limsup(x_n+y_n) < \limsup x_n + \limsup y_n$. Notice that *strict* inequality is required for this example.

9. Give an example of a decreasing nest of finite open intervals $(a_1, b_1) \supset (a_2, b_2) \supset \cdots \supset (a_n, b_n) \supset \cdots$ for which $\displaystyle\bigcap_{n=1}^{\infty}(a_n, b_n) = \emptyset$.

10. Find an open cover $\mathcal{O}$ of the real line that has no finite subcover.

11. True or False: Every open cover of a finite[1] set has a finite subcover.

[1] A *finite* set is a set with only finitely many elements.

12. Let $E = \mathbb{R} \setminus \{p\}$, a *punctured* real line with the point p missing. Find an open cover of E that has no finite subcover.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Suppose $a \le x_n \le b$ for all n and suppose further that $x_n \to L$. Prove: $L \in [a, b]$. (Hint: If $L < a$ or if $L > b$, obtain a contradiction. Be sure to show that you understand limits in terms of $\epsilon > 0$ and $N \in \mathbb{N}$.)

B. Denote the nth tail of the sequence $x_n \in \mathbb{R}$ by $T_n = \{x_j \mid j \ge n\}$. Suppose the following special condition is satisfied: For each $n \in \mathbb{N}$, T_n has a smallest element. Prove that there exists a monotone increasing subsequence, $x_{n_j} \nearrow$, indexed by j.

C. Let $E = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$. *Find* an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E that has no finite subcover. *Prove* that $\mathcal{O}$ is an open cover of E and that $\mathcal{O}$ has no finite subcover.

Solutions and Class Statistics

1. For example, $(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \neq \vec{i} \times (\vec{i} \times \vec{j}) = -\vec{j}$. Learning the right-hand rule for vector products makes this very, very easy. It can be done correctly with determinants, but that is a *disgusting* waste of effort. Remark: A vector in $\mathbb{R}^3$ is not a 3×3 matrix! Also, $(1, 0) \notin \mathbb{R}^3$.

2. Counterexample: If $x_n = (-1)^n = y_n$, then $x_n y_n$ converges although each factor diverges. It is understood that your sequence must be a sequence of real numbers!

3. For example, let $S = (0, 1)$, so that $\sup(s) = 1 \notin S$.

4. True, because of the Bolzano-Weierstrass Theorem.

5. False: see for example Exercise 1.24(d).

6. Counterexample: Let $x_n = (-1)^n = y_n$.

7. $\limsup x_n = 1$ and $\liminf x_n = -1$.

8. Example: let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$.

9. For example, let $(a_n, b_n) = \left(0, \dfrac{1}{n}\right)$.

10. For example, let $\mathcal{O} = \{(-n, n) \mid n \in \mathbb{N}\}$. Remember that the *union* of the elements of an open cover is just *one* open set, **not** the open cover itself!

11. True.

12. For example, $\mathcal{O} = \left\{O_n = \left(-\infty, p - \dfrac{1}{n}\right) \cup \left(p + \dfrac{1}{n}, \infty\right) |n \in \mathbb{N}\right\}$.

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs.

In proof (A), it is important to use quantifiers correctly. We need to use the concept of convergence correctly, expressed in terms of inequalities that need to be satisfied. You need to specify how the choice of n implies the desired inequalities. Specifically, if $L < a$ you need to choose an $\epsilon > 0$ expressed in terms of a and L. This is easy to do: See the proof provided in class!

In proof (B), one needs to make sure that the monotone increasing x_{n_j} subsequence of x_n has the indices n_j strictly increasing. So it matters how one chooses the tails in which to take a smallest element.

In proof (C), it is essential that your open cover actually cover E. Please work hard at writing mathematics correctly. This will help to clarify your thinking. A jumble of non-sequiturs is not a logical proof of anything.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	6				
80-89 (B)	2				
70-79 (C)	5				
60-69 (D)	1				
0-59 (F)	7				
Test Avg	73.5%	%	%	%	%
HW Avg	8.26				
HW/Test Correl	0.90	–			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{22}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.